Computability Theory and Infinitary Combinatorics

Noah A. Hughes
Winston-Salem State University
hughesna@wssu.edu — nahughes.github.io

Friday, December 2nd, 2022

Fall 2022 Colloquium Series
Department of Mathematical Sciences
Appalachian State University

Before I begin speaking,

I'd like to say thanks.

–Credit to Raymond Smullyan for the witticism.

Motivation — Mathematical Logic

From Paul Shafer:

I describe calculus as the mathematics of change[,]
geometry as the mathematics of shape,
...[and math] logic as the mathematics of mathematics.

We take mathematical theorems, proofs, and constructions as our objects of study, specifically from infinitary[1] combinatorics.

The goal is to understand the foundational mechanics of mathematics: e.g.,

- discern underlying connections in seemingly disparate mathematical theorems
- determine the necessary ingredients in any proof of a particular theorem.

Computability theory provides a powerful viewpoint from which to conduct this analysis.

[1] countably infinite

Two Frameworks

Reverse Mathematics

Prove results of the form:

Over a weak base theory $\mathcal{B}$, the axiom A is both necessary and sufficient to prove the familiar theorem ξ.

$$\mathcal{B} \vdash A \iff \xi$$

Computability Theoretic Reductions

Reduce the problem of finding one desired mathematical object to finding another object by use of a computable transformation.

$$\begin{array}{ccc}
X & -\,\Phi \rightarrow & \widehat{X} \\
\textbf{commutative ring, } R & & \textbf{infinite binary tree, } T \\
\vdots & & \vdots \\
Y & \leftarrow \Psi\, - & \widehat{Y} \\
\textbf{prime ideal, } P & & \textbf{infinite path, } f
\end{array}$$

Two Frameworks

Reverse Mathematics

Prove results of the form:

Over a weak base theory $\mathcal{B}$, the axiom A is both necessary and sufficient to prove the familiar theorem ξ.

ZF set theory $\vdash$ Axiom of Choice $\iff$ Zorn's Lemma

Computability Theoretic Reductions

Reduce the problem of finding one desired mathematical object to finding another object by use of a computable transformation.

$$\begin{array}{ccc}
X & -\,\Phi\,\rightarrow & \widehat{X} \\
\textbf{commutative ring, } R & & \textbf{infinite binary tree, } T \\
\vdots & & \vdots \\
Y & \leftarrow\,\Psi\,- & \widehat{Y} \\
\textbf{prime ideal, } P & & \textbf{infinite path, } f
\end{array}$$

Essentials of Computability Theory

Fix an effective enumeration of the partial computable functions on $\mathbb{N}$

$$\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_e, \Phi_{e+1}, \ldots$$

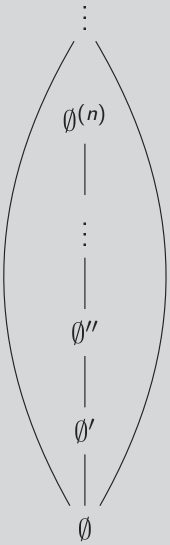
A set C is *computable* if its characteristic function $\chi_C = \Phi_e$ for some e.

The set $\emptyset' = \{e : \Phi_e(e) \text{ halts }\}$ is the canonical noncomputable set.

Given a set $A \subseteq \mathbb{N}$, we *relativize* each Φ_e to Φ_e^A.

Another set B is *A-computable* if $\chi_B = \Phi_e^A$ for some e.

The set $A' = \{e : \Phi_e^A(e) \text{ halts}\}$ is the canonical set that A cannot compute. We call A' the *Turing jump* of A.



Computable Transformations

We can view each **total** computable function Φ as a *functional* on subsets of $\mathbb{N}$ or $2^{\mathbb{N}}$.

$\boxed{\text{Ex}}$ Suppose $\Phi^A = \chi_B$ with

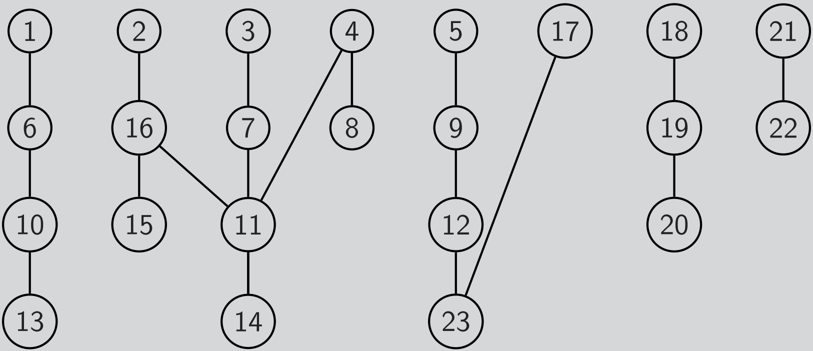
$$A = \{2k : k \in \mathbb{Z}\} \text{ and } B = \{p \in \mathbb{Z} : p \text{ is prime}\}$$

$$\begin{array}{c} A : \langle 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots \rangle \\ | \\ \Phi \\ \downarrow \\ B : \langle 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, \dots \rangle \end{array}$$

In this way, we can transform a countable ring R (coded in $\mathbb{N}$) to an infinite binary tree $T = \Phi^R$ (coded in $\mathbb{N}$).

Interregnum — A combinatorial preview

Posets and Graphs



$13 \leq 10 \leq 6 \leq 1$; $\quad 15 \leq 16 \leq 2$; $\quad$ 15 and 11 are incomparable ...

Reverse Mathematics with the Big Five

Goal: Determine exactly which axiom(s) are needed in the proof of a (countable analogue of a) particular theorem.

Method: Given a theorem ξ, fix a weak base axiom system $\mathcal{B}$ which cannot prove ξ and find an additional axiom A such that:

$\mathcal{B} \vdash \mathsf{A} \to \xi$	$\mathcal{B} \vdash \xi \to \mathsf{A}$
a "regular" proof	a "reversed" proof
$\mathsf{ZF} \vdash \mathsf{AC} \to \mathsf{ZL}$	$\mathsf{ZF} \vdash \mathsf{ZL} \to \mathsf{AC}$
a "regular" proof	a "reversed" proof

Traditionally, we use RCA_0 as the base system and require *only four additional set existence axioms* to conduct this analysis.

The Big Five Phenomenon

The Big Five subsystems of Z_2

RCA_0: Axioms for first-order arithmetic, a weak induction principle, and the "recursive comprehension axiom."

Algorithmically definable sets exist.

WKL_0: RCA_0 + weak Kőnig's lemma

Every infinite binary tree has an infinite path.

ACA_0: RCA_0 + "arithmetical comprehension axiom"

Sets definable with number quantifiers exist.

ATR_0: ACA_0 + (axioms for) "arithmetical transfinite recursion"

Sets definable by recursion on a given countable well-order exist.

Π^1_1-CA_0: ACA_0 + "Π^1_1 comprehension axiom"

Sets definable with one universal set quantifier exist.

The Big Five Phenomenon in Algebra

$$
\begin{array}{rcl}
\mathsf{RCA}_{\mathbf{0}} & \vdash & \text{Every countable field has an algebraic closure.} \\
\top & & \\
\mathsf{WKL}_{\mathbf{0}} & \leftrightarrow & \text{Every countable commutative ring with identity} \\
 & & \quad \text{has a prime ideal.} \\
\mathsf{ACA}_{\mathbf{0}} & \leftrightarrow & \text{Every countable commutative ring with identity} \\
 & & \quad \text{has a maximal ideal.} \\
\mathsf{ATR}_0 & \leftrightarrow & \text{Ulm's theorem: Any two countable reduced Abelian} \\
 & & p\text{-groups which the same Ulm invariants are isomorphic.} \\
\Pi^1_1\text{-}\mathsf{CA}_0 & \leftrightarrow & \text{Every countable Abelian group is the direct sum of} \\
 & & \quad \text{a divisible group and a reduced group.}
\end{array}
$$

The Big Five Phenomenon in Analysis

RCA_0	$\vdash$	The intermediate value theorem.
$\top$		
WKL_0	$\leftrightarrow$	Heine/Borel: every covering of $[0,1]$ with a sequence of open intervals has a finite subcovering.
ACA_0	$\leftrightarrow$	Balzano/Weierstraß: every bounded sequence of real numbers has a convergent subsequence.
ATR_0	$\leftrightarrow$	Perfect set theorem: every uncountable closed set has a perfect subset.
Π^1_1-CA_0	$\leftrightarrow$	Cantor/Bendixson: Every closed subset of $\mathbb{R}^n$ is the union of a countable set and a perfect set.

The Big Five Phenomenon in Combinatorics

$$\begin{array}{rcl}
\mathsf{RCA}_{\mathbf{0}} & \vdash & \text{Every finite bipartite graph satisfying Hall's condition} \\
\top & & \quad\text{has a perfect matching.} \\
\mathsf{WKL}_{\mathbf{0}} & \leftrightarrow & \text{Every infinite 2-branching tree has an infinite path.} \\
\mathsf{ACA}_{\mathbf{0}} & \leftrightarrow & \text{Kőnig's lemma: every infinite finitely-branching tree} \\
& & \quad\text{has an infinite path.} \\
\mathsf{ATR}_0 & \leftrightarrow & \text{Any two countable well-orders are comparable.} \\
\Pi^1_1\text{-}\mathsf{CA}_0 & \leftrightarrow & \text{Every tree has a largest perfect subtree.}
\end{array}$$

Graphs and hypergraphs in reverse mathematics

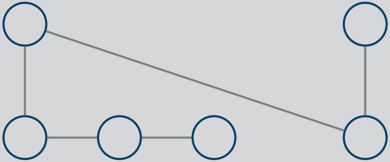
A *hypergraph* $H = (V, E)$ consists of a set $V \subseteq \mathbb{N}$ of vertices and a collection $E = \{e_0, e_1, e_2, \dots\} \subseteq \mathcal{P}(V)$ of edges.
We say $u, v \in V$ are *adjacent* if $u, v \in e$ for some edge $e \in E$.

A *graph* $G = (V, E)$ is a hypergraph in which every edge has size 2.

A graph is *bipartite* if $V = X \sqcup Y$ and for every $e \in E$, we have $e \not\subseteq X$ and $e \not\subseteq Y$.

A *proper k-coloring* of a (hyper)graph is a map $c : V \to \{0, 1, \dots, k-1\}$ which is nonconstant on every edge $e \in E$.

Fact: A graph G is bipartite if and only if there exists a proper 2-coloring of G.



Graphs and hypergraphs in reverse mathematics

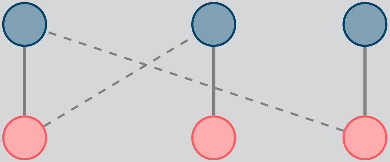
A *hypergraph* $H = (V, E)$ consists of a set $V \subseteq \mathbb{N}$ of vertices and a collection $E = \{e_0, e_1, e_2, \dots\} \subseteq \mathcal{P}(V)$ of edges.
We say $u, v \in V$ are *adjacent* if $u, v \in e$ for some edge $e \in E$.

A *graph* $G = (V, E)$ is a hypergraph in which every edge has size 2.

A graph is *bipartite* if $V = X \sqcup Y$ and for every $e \in E$, we have $e \not\subseteq X$ and $e \not\subseteq Y$.

A *proper k-coloring* of a (hyper)graph is a map $c : V \to \{0, 1, \dots, k-1\}$ which is nonconstant on every edge $e \in E$.

Fact: A graph G is bipartite if and only if there exists a proper 2-coloring of G.



Unique matchings

A *matching* of a bipartite graph $G = (X \cup Y, E)$ is an injection $f\colon X \to Y$ such that $\{x, f(x)\} \in E$ for all $x \in X$.

We use $A(x)$ to denote the set of vertices adjacent to x: $A(x) = \{y : \{x, y\} \in E\}$. The size of $A(x)$ is the *degree* of x.

A graph $G = (V, E)$ is *locally finite* if $A(x)$ is finite for all $x \in V$.

Theorem (with Jeff Hirst)

A locally finite bipartite graph $G = (X \cup Y, E)$ has a unique matching if and only if there is an enumeration $\langle x_i \rangle_{i \in \mathbb{N}}$ of the vertices in X such that for all n:

$$|A(x_0, x_1, \ldots, x_n)| = n + 1.$$

Theorem

A bipartite graph $G = (X \cup Y, E)$ has a unique matching if and only if there is a well-order $(X, \preceq)$ such that for every $x \in X$ there is a unique $y \in Y$ with

$$A(x) - A(\{x' : x' \prec x\}) = \{y\}.$$

Unique matchings

Theorem (with Jeff Hirst)

A locally finite bipartite graph $G = (X \cup Y, E)$ has a unique matching if and only if there is an enumeration $\langle x_i \rangle_{i \in \mathbb{N}}$ of the vertices in X such that for all n:

$$|A(x_0, x_1, \ldots, x_n)| = n + 1.$$

Let MTE denote the "only if" direction and ETM denote the "if" direction.

Theorem (H.)

A bipartite graph $G = (X \cup Y, E)$ has a unique matching if and only if there is a well-order $(X, \leq_X)$ such that for every $x \in X$ there is a unique $y \in Y$ with

$$A(x) - A(\{x' : x' \prec x\}) = \{y\}.$$

Let MTO denote the "only if" direction and OTM denote the "if" direction.

Unique matchings and reverse mathematics

The enumeration provides a coding advantage when proving reversals.

Theorem (with Jeff Hirst)

Over RCA_0,

1. *the statement* ETM *is provable; and*
2. *the statement* MTE *is provably equivalent to* ACA_0.

$$
\begin{tikzcd}
\mathsf{ACA}_0 \arrow[r, leftrightarrow] \arrow[dd] \arrow[dr] & \mathsf{OTM} \arrow[r, leftrightarrow] & \mathsf{MTE} \\
& \mathsf{MTO} \arrow[ul, bend left, "?"] \arrow[dl] & \\
\mathsf{WKL}_0 \arrow[d] \arrow[ur, bend right, "/" marking] & & \\
\mathsf{RCA}_0 \arrow[rr] & & \mathsf{ETM}
\end{tikzcd}
$$

Theorem (H.)

1. *Over* RCA_0, *the statement* OTM *is provably equivalent to* ACA_0*; and*
2. *the statement* MTO *is provable in* ACA_0.
3. *the statement* MTO *is not provable in* WKL_0.

Picture #1 of Reverse Math: The Big Five

$$\begin{tikzcd}
\Pi^1_1\text{-}\mathsf{CA}_0 \arrow[d] & \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[r, leftrightarrow] \arrow[d] \arrow[dr] & \mathsf{RT}^{n\geq 3}_{k\geq 2} \\
\mathsf{WKL}_0 \arrow[d] & \mathsf{RT}^2_2 \arrow[dl] \\
\mathsf{RCA}_0 &
\end{tikzcd}$$

Ramsey's Theorem

Let $[\mathbb{N}]^n$ denote the set of all n-element subsets of $\mathbb{N}$.

Call $c : [\mathbb{N}]^n \to k = \{0, 1, 2, \ldots, k-1\}$ a *k-coloring* of $[\mathbb{N}]^n$.

$\mathbf{RT^n_k}$: Every k-coloring of $[\mathbb{N}]^k$ has an infinite homogeneous set H.

$\boxed{\text{Ex}}$ ▶ RT^1_2 is the Pigeonhole Principle.

▶ RT^2_2 states every infinite graph contains an infinite clique or anticlique.

Note: the combinatorial subtleties of RT^n_k collapse for $n \geq 3$ and $k \geq 2$.

Picture #2 of Reverse Math: The Zoo

$$
\begin{tikzcd}
\Pi^1_1\text{-}\mathsf{CA}_0 \arrow[d] & \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[r, leftrightarrow] \arrow[dd] \arrow[dr] & \mathsf{RT}^{n\geq 3}_{k\geq 2} \\
 & \mathsf{RT}^2_2 \arrow[d] \\
\mathsf{WKL}_0 \arrow[r, dashed, shift left] \arrow[d] & \textbf{The Zoo} \arrow[l, dashed, shift left] \arrow[dl] \\
\mathsf{RCA}_0 &
\end{tikzcd}
$$

Ramsey's Theorem

Let $[\mathbb{N}]^n$ denote the set of all n-element subsets of $\mathbb{N}$.

Call $c : [\mathbb{N}]^n \to k = \{0, 1, 2, \dots, k-1\}$ a *k-coloring* of $[\mathbb{N}]^n$.

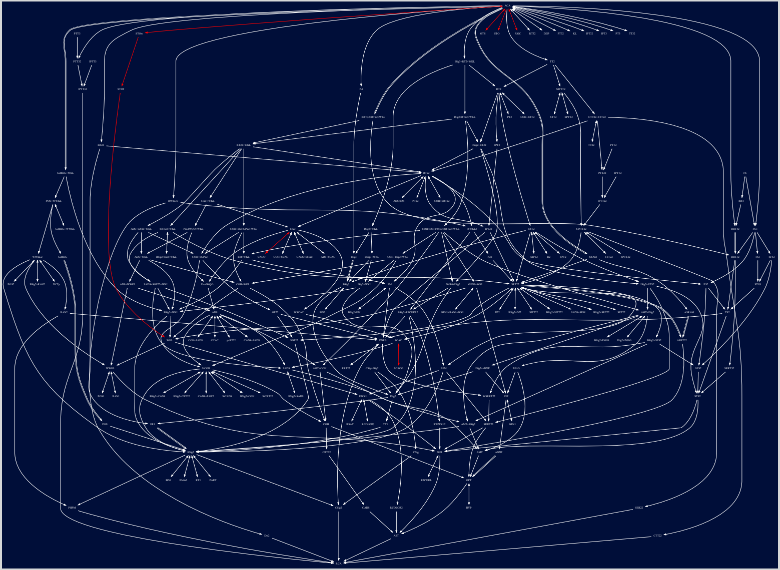
$\mathbf{RT^n_k}$: Every k-coloring of $[\mathbb{N}]^k$ has an infinite homogeneous set H.

$\boxed{\text{Ex}}$ ▶ RT^1_2 is the Pigeonhole Principle.

▶ RT^2_2 states every infinite graph contains an infinite clique or anticlique.

Note: the combinatorial subtleties of RT^n_k collapse for $n \geq 3$ and $k \geq 2$.

Picture #2 of Reverse Math: The Zoo



Computability Theoretic Reductions

We recast a mathematical theorem as a *formal problem* P made up of instances X and associated solutions Y.

For example:

- MI: The problem whose instances R are countable rings with identity with solutions M which are maximal ideals in R.
- RT^3_2: The problem whose instances c are 2-colorings of $[\mathbb{N}]^3$ with solutions H which are homogeneous with respect to c.

Problems	**P**	$\leq_{\mathrm{sW}}$	**Q**
Instances	X **commutative ring,** R	$- \Phi \rightarrow$	$\widehat{X}$ **2-coloring of** $[\mathbb{N}]^3$**,** c
	$\vdots$		$\vdots$
Solutions	Y **maximal ideal,** M	$\leftarrow \Psi -$	$\widehat{Y}$ **homogeneous set,** H

Four Reductions — Uniform vs. Non-uniform

$$\begin{tikzcd}
\mathsf{P} \arrow[r, phantom, "\leq_{\mathrm{sW}}"] & \mathsf{Q} \\
X \arrow[r, "\Phi"] \arrow[d, dotted] & \widehat{X} \arrow[d, dotted] \\
Y & \widehat{Y} \arrow[l, "\Psi^{\widehat{Y}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{P} \arrow[r, phantom, "\leq_{\mathrm{W}}"] & \mathsf{Q} \\
X \arrow[r, "\Phi"] \arrow[d, dotted] & \widehat{X} \arrow[d, dotted] \\
Y & \widehat{Y} \arrow[l, "\Psi^{X \oplus \widehat{Y}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{P} \arrow[r, phantom, "\leq_{\mathrm{sc}}"] & \mathsf{Q} \\
X \arrow[r, "\text{computes}"] \arrow[d, dotted] & \widehat{X} \arrow[d, dotted] \\
Y & \widehat{Y} \arrow[l, "\widehat{Y}\ \text{computes}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{P} \arrow[r, phantom, "\leq_{\mathrm{c}}"] & \mathsf{Q} \\
X \arrow[r, "\text{computes}"] \arrow[d, dotted] & \widehat{X} \arrow[d, dotted] \\
Y & \widehat{Y} \arrow[l, "X \oplus \widehat{Y}\ \text{computes}"']
\end{tikzcd}$$

A finer analysis

Seemingly different combinatorial principles that are equivalent in the reverse mathematics setting can often be distinguished using these reductions.

Showing P is *not* reducible to Q under one of these reductions reveals precise differences that reverse mathematics may obfuscate,

e.g. whether or not there is a uniform proof of P from Q.

$$\begin{tikzcd}
 & \mathsf{P} \leq_{\mathsf{sW}} \mathsf{Q} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathsf{P} \leq_{\mathsf{sc}} \mathsf{Q} \arrow[dr, Rightarrow] & & \mathsf{P} \leq_{\mathsf{W}} \mathsf{Q} \arrow[dl, Rightarrow] \\
 & \mathsf{P} \leq_{\mathsf{c}} \mathsf{Q} \arrow[d, dashed] & \\
 & \mathsf{RCA} \vdash \mathsf{Q} \to \mathsf{P} &
\end{tikzcd}$$

$\mathsf{RCA}_0 \vdash \mathsf{RT}^3_2 \leftrightarrow \mathsf{RT}^4_2$ $\qquad$ $\mathsf{RT}^3_2 \not\leq_{\mathsf{W}} \mathsf{RT}^4_2$

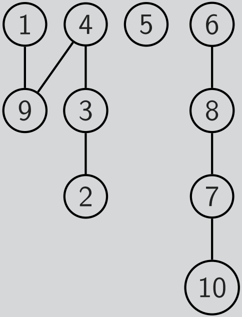
$\mathsf{RCA}_0 \vdash \mathsf{RT}^n_j \leftrightarrow \mathsf{RT}^n_k$, $n \geq 1$, $k > j \geq 2$ $\qquad$ $\mathsf{RT}^n_j \not\leq_{\mathsf{W}} \mathsf{RT}^n_k$

Posets and Dilworth's Theorem

A partially ordered set (*poset*) is a pair $(P, \leq_P)$ where $\leq_P$ is a reflexive, transitive, and antisymmetric relation on $P \subseteq \omega$.

We say $X \subseteq P$ is:

- a *chain* if for all $x, y \in X$, either $x \leq_P y$ or $y \leq_P x$.
- an *antichain* if for all $x, y \in X$, we have $x \not\leq_P y$ and $y \not\leq_P x$.



Theorem (Dilworth's theorem)

In any finite poset $(P, \leq_P)$, the size of the largest antichain equals the minimum number of chains needed to cover P.

Theorem (CAC: Chain-Antichain Theorem)

If $(P, \leq_P)$ is an infinite poset, then P contains either an infinite chain or an infinite antichain.

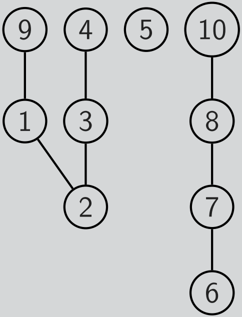
Stable and ω-ordered posets

We say an element $x \in P$ is

- *small* if $x \leq_P y$ for all but finitely many $y \in P$
- *large* if $y \leq_P x$ for all but finitely many $y \in P$
- *isolated* if $x \,|_P\, y$ for all but finitely many $y \in P$

We say a poset $(P, \leq_P)$ is

- *stable* if every $x \in P$ is either small or isolated, or every $x \in P$ is either large or isolated.
- *ω-ordered* if for all $x, y \in P$, we have $x \leq_P y$ implies $x \leq y$ in ω.



Let SCAC be the restriction of CAC to stable posets;
CAC$^{\text{ord}}$ be the restriction of CAC to ω-ordered posets; and
SCAC$^{\text{ord}}$ be the restriction of CAC to stable ω-ordered posets.

Chains and antichains

Theorem (Hirschfeldt and Shore)

$\mathsf{RCA}_0 \vdash \mathsf{CAC}$ *strictly implies* SCAC.

Theorem (H.)

- $\mathsf{RCA}_0 \vdash \mathsf{CAC}^{\mathrm{ord}} \leftrightarrow \mathsf{CAC}$
- $\mathsf{RCA}_0 \vdash \mathsf{SCAC}^{\mathrm{ord}} \leftrightarrow \mathsf{SCAC}$

Coro: $\mathsf{RCA}_0 \vdash \mathsf{CAC}^{\mathrm{ord}}$ *strictly implies* $\mathsf{SCAC}^{\mathrm{ord}}$.

Theorem (H.)

- $\mathsf{CAC}^{\mathrm{ord}} \not\leq_{c} \mathsf{CAC}$
- $\mathsf{SCAC}^{\mathrm{ord}} \equiv_{c} \mathsf{SCAC}$
- $\mathsf{SCAC}^{\mathrm{ord}} \not\leq_{W} \mathsf{SCAC}$
- $\mathsf{SCAC}^{\mathrm{ord}} \not\leq_{sc} \mathsf{SCAC}$

$$\begin{tikzcd}
& P \leq_{sW} Q \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
P \leq_{sc} Q \arrow[dr, Rightarrow] & & P \leq_{W} Q \arrow[dl, Rightarrow] \\
& P \leq_{c} Q \arrow[d, dashed] & \\
& \mathsf{RCA} \vdash Q \to P &
\end{tikzcd}$$

There's always more math to do.

- Separate equivalent unique matching principles with $\leq_{\mathsf{sW}}$, etc.
- Find the exact location of MTO in the reverse mathematics Zoo
- Compare ω-ordered posets and *weakly stable* posets.
- Analyze more mathematics within these frameworks.
- Analyze the frameworks themselves!

References

[1] Jeffry L. Hirst and Noah A. Hughes, *Reverse mathematics and marriage problems with unique solutions.*, Arch. Math. Logic **54** (2015), 49-57. DOI 10.1007/s00153-014-0401-z.

[2] ———, *Reverse mathematics and marriage problems with finitely many solutions.*, Arch. Math. Logic **55** (2016), 1015-1024. DOI 10.1007/s00153-016-0509-4.

[3] Noah A. Hughes, *Applications of Computability Theory to Infinitary Combinatorics*, PhD Thesis, University of Connecticut (2021).

---

[4] Denis R. Hirschfeldt, *Slicing the truth*, Lecture Notes Series. Institute for Mathematical Sciences. National University of Singapore, vol. 28, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. MR3244278

[5] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. DOI 10.1017/CBO9780511581007, MR2517689.

Thank you for your attention!

Unique colorings of graphs and hypergraphs - Bonus #1

Davis, Hirst, Pardo, and Ransom showed that there is no arithmetic way to determine if a hypergraph has a proper k-coloring.

Theorem (Davis, Hirst, Pardo, and Ransom)

Over RCA_0*, the following statement is equivalent to* Π^1_1-CA_0*:*

- *If* $\langle H_i \rangle_{i \in \mathbb{N}}$ *is a sequence of hypergraphs then there is a function* $f : \mathbb{N} \to \{0, 1\}$ *such that* $f(i) = 1$ *if and only if* H_i *has a proper* k*-coloring.*

Theorem (H.)

Over RCA_0*, the following statement is equivalent to* ATR_0*:*

- *If* $\langle H_i \rangle_{i \in \mathbb{N}}$ *is a sequence of hypergraphs which admit at most one proper 2-coloring then there is a function* $f : \mathbb{N} \to \{0, 1\}$ *such that* $f(i) = 1$ *if and only if* H_i *has a proper 2-coloring.*

Unique colorings of graphs and hypergraphs - Bonus #1

Theorem (with Jeff Hirst)

Over RCA_0*, the following statement is equivalent to* ACA_0*:*

- *If* $\langle G_i \rangle_{i \in \mathbb{N}}$ *is a sequence of graphs then there is a function* $s : \mathbb{N} \to \{0, 1, 2\}$ *such that*

$$s(i) = \begin{cases} 0 & \text{if } G_i \text{ has no proper 2-coloring} \\ 1 & \text{if } G_i \text{ has a unique proper 2-coloring} \\ 2 & \text{if } G_i \text{ has many proper 2-colorings} \end{cases} \qquad (*)$$

Theorem (with Jeff Hirst)

Over RCA_0*, the following statement is equivalent to* Π^1_1-CA_0*:*

- *If* $\langle H_i \rangle_{i \in \mathbb{N}}$ *is a sequence of hypergraphs then there is a function* $s : \mathbb{N} \to \{0, 1, 2\}$ *satisfying* $(*)$.

Unique matchings and reverse mathematics — Bonus #2

The enumeration provides a large advantage in the coding potential of these principles.

Theorem (with Jeff Hirst)

Over RCA_0,

1. *the statement* ETM *is provable; and*
2. *the statement* MTE *is provably equivalent to* ACA_0*.*

Theorem (H.)

1. *Over* RCA_0*, the statement* OTM *is provably equivalent to* ACA_0*; and*
2. *the statement* MTO *is provable in* ACA_0*.*
3. *the statement* MTO *is not provable in*

$$
\begin{tikzcd}
\mathrm{ACA}_0 \arrow[r, leftrightarrow] \arrow[dr] \arrow[ddd] & \mathrm{OTM} \arrow[r, leftrightarrow] & \mathrm{MTE} \\
& \mathrm{MTO} \arrow[ul, bend left, "?"] \arrow[d] & \\
& \mathrm{Ext}(\omega^*) \arrow[uul, bend right, "?"'] \arrow[dl] & \\
\mathrm{WKL}_0 \arrow[ur, bend right, "/" marking] \arrow[d] & & \\
\mathrm{RCA}_0 \arrow[rr] & & \mathrm{ETM}
\end{tikzcd}
$$