On generalizations of Hall's theorem

Noah A. Hughes
noah.hughes@uconn.edu
University of Connecticut

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NERDS 14.0

Reverse mathematics.

Goal: Determine exactly which set existence axioms are needed in the proof of a (countable analogue) of a familiar theorem.

Method: Prove results of the form

$$\mathsf{RCA}_0 \vdash \mathbf{Ax} \leftrightarrow \mathbf{Thm}$$

where the base system used is

$$\mathsf{RCA}_0 : \begin{cases} \text{axioms of second order arithmetic} \\ \text{with induction restricted to } \Sigma^0_1 \text{ formulas} \\ \text{and comprehension restricted to } \Delta^0_1 \text{ formulas} \end{cases}$$

The "big five" subsystems.

RCA_0

$\Downarrow$

WKL_0: $\mathsf{RCA}_0 +$ "every infinite binary tree has an infinite pa

$\Downarrow$

ACA_0: $\mathsf{RCA}_0 +$ comprehension for arithmetical formulas

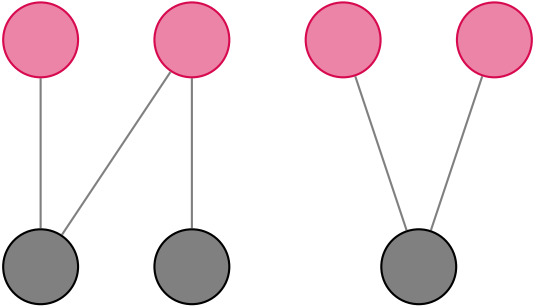
$\Downarrow$

ATR_0: $\mathsf{RCA}_0 +$ iterability of arithmetical operators along any well-order

$\Downarrow$

$\Pi^1_1{-}\mathsf{CA}_0$: $\mathsf{RCA}_0 +$ comprehension for Π^1_1 formulas

Matchings.



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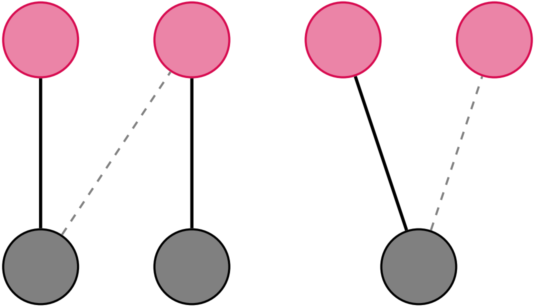

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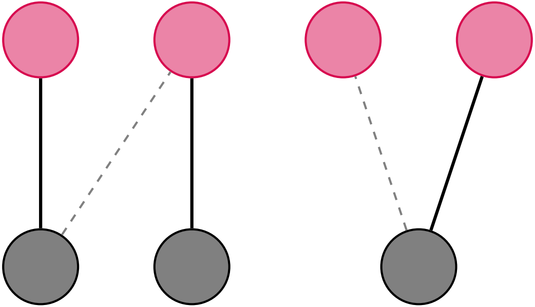
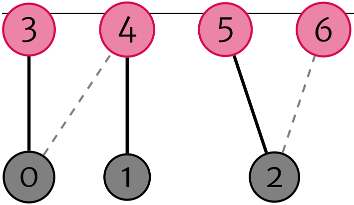


Formalization.

A matching problem is a triple $P = (A, B, R)$ where $A, B \subseteq \mathbb{N}$ and $R \subseteq A \times B$.

If $(a, b) \in R$ we say b is a permissable match of a and set $R(a) = \{b : (a, b) \in R\}$.

A solution to a matching problem is an injection $f : A \to B$ such that $f(a) \in R(a)$ for all $a \in A$.



$A = \{0, 1, 2\}$

$B = \{3, 4, 5, 6\}$

The Halls' theorems.

Theorem (Philip Hall)

Let $P = (A, B, R)$ be a matching problem in which A is finite and every element has finitely many permissable matches. If $|A_0| \leq |R(A_0)|$ for every $A_0 \subseteq A$, then P has a solution.

Theorem (Marshall Hall)

Let $P = (A, B, R)$ be a matching problem in which every element has finitely many permissable matches. If $|A_0| \leq |R(A_0)|$ for every $A_0 \subseteq A$, then P has a solution.

Theorem (Hirst)

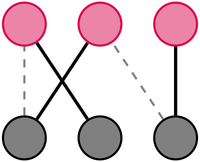
The following are provable in RCA_0

1. *Philip Hall's theorem*
2. $\mathsf{ACA}_0 \leftrightarrow$ *Marshall Hall's theorem*

Uniqueness.

Theorem (Hirst, Hughes)

A matching problem $P = (A, B, R)$, in which every element has finitely many permissable matches, has a unique solution if and only if there is an enumeration of A, say $\langle a_i \rangle_{i \geq 1}$ such that for every $n \geq 1$, $|R(a_1, a_2, \ldots, a_n)| = n$.



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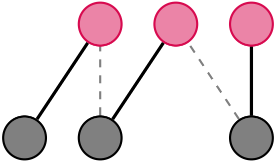

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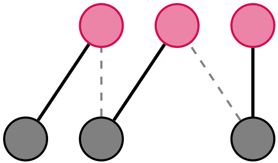

Theorem (Hirst, Hughes)
Over RCA_0*, the following are equivalent*

1. ACA_0
2. *The above theorem*

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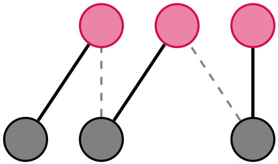

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Over RCA_0, the following are equivalent

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A generalization.

We now consider arbitrary (countable) matching problems in which any element may have infinitely many permissable matches.

Theorem

A matching problem $P = (A, B, R)$ has a unique solution if and only if there is a well-order $(A, <_A)$ such that for each $a \in A$, there is a unique $b \in B$ satisfying

$$R(a) - R(\{a' : a' <_A a\}) = \{b\}.$$

For convenience we label the forward direction STO and the reverse direction OTS.

Conjecture (Hirst)

Over RCA_0

1. ATR_0 *is provably equivalent to* STO
2. *and* ACA_0 *is provably equivalent to* OTS.

Current results.

Theorem (Hughes)

Over RCA_0*, the following are equivalent*

1. ACA_0
2. OTS*: A matching problem* $P = (A, B, R)$ *has a unique solution if there is a well-order* $(A, <_A)$ *such that for each* $a \in A$*, there is a unique* $b \in B$ *satisfying*

$$R(a) - R(\{a' : a' <_A a\}) = \{b\}.$$

Theorem (Hughes)

The following is provable in ATR_0*:*

> STO*: A matching problem* $P = (A, B, R)$ *has a unique solution only if there is a well-order* $(A, <_A)$ *such that for each* $a \in A$*, there is a unique* $b \in B$ *satisfying* $R(a) - R(\{a' : a' <_A a\}) = \{b\}$.

ATR_0 proves STO: a sketch.

Fix a matching problem $P = (A, B, R)$ with unique solution f.

Our goal is to build a well order such that each element has exactly one permissable match that it's predeccesors do not have.

Given an initial segment $(A_0, \leq)$ of the desired well order $(A, \leq)$, it is arithmetical to find a suitable next element:

$$\psi(A_0, a) : R(a) - \bigcup_{a' \in A_0} R(a') = \{f(a)\}.$$

Thus, in ATR_0, we may iteratively construct the desired well order by applying ψ at each stage to find an appopriate $a \in A$ to append to the order.

We need only determine which well order to iterate upon.

Use a short tree.

Recall for a given tree T, the Kleene–Brouwer order $\mathrm{KB}(T)$ is such that

$$\sigma <_{\mathrm{KB}} \tau \iff \sigma \succ \tau \vee \exists n(\sigma \upharpoonright n = \tau \upharpoonright n \wedge \sigma(n) < \tau(n))$$

ACA_0 suffices to show the Kleene–Brouwer order of a well-founded tree is a well-order.

We construct a well-founded tree T which encodes the dependencies of elements of A and iterate upon $\mathrm{KB}(T)$.

Let

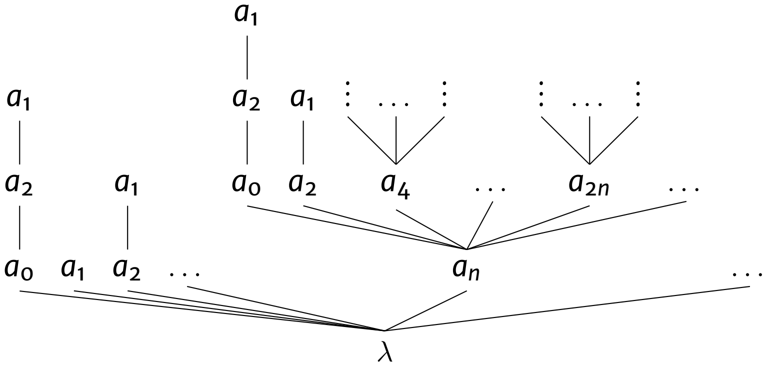
$$\begin{aligned} T_0 &= \langle\rangle \cup \{\langle a\rangle : a \in A\} \\ T_{s+1} &= T_s \cup \{\sigma^\frown\langle a\rangle : \sigma \in T_s, a \neq \sigma(|\sigma|-1), f(a) \in R(\sigma(|\sigma|-1))\} \end{aligned}$$

And set $T = \cup_{s\in\omega} T_s$.

An example.

The unique solution of P guarentees T is well-founded.

$$R(a_0) = \{f(a_0), f(a_2)\}, \quad R(a_1) = \{f(a_1)\}, \quad R(a_2) = \{f(a_2), f(a_1)\},$$
$$\text{and } R(a_n) = \{f(a_n)\} \cup \{f(a_{2i}) : i \in \omega\}$$



An example.

$$
\begin{array}{r}
\langle a_0, a_2, a_1\rangle < \langle a_0, a_2\rangle < \langle a_0\rangle \\
\wedge \\
\langle a_2, a_1\rangle < \langle a_2\rangle \\
\wedge \\
\vdots \\
\wedge \\
\langle a_n, a_0, a_2, a_1\rangle < \langle a_n, a_0, a_2\rangle < \langle a_n, a_0\rangle < \langle a_n, a_2, a_1\rangle \\
< \langle a_n, a_2\rangle < \cdots < \langle a_n, a_4\rangle < \cdots < \langle a_n, a_{2n}\rangle < \cdots < \langle a_n\rangle \\
\wedge \\
\langle a_{n+1}\rangle \\
\wedge \\
\vdots
\end{array}
$$

An example.

$$\begin{array}{r}
\langle a_0, a_2, a_1\rangle < \langle a_0, a_2\rangle < \langle a_0\rangle \\
\wedge \\
\langle a_2, a_1\rangle < \langle a_2\rangle \\
\wedge \\
\vdots \\
\wedge \\
\langle a_n, a_0, a_2, a_1\rangle < \langle a_n, a_0, a_2\rangle < \langle a_n, a_0\rangle < \langle a_n, a_2, a_1\rangle \\
< \langle a_n, a_2\rangle < \cdots < \langle a_n, a_4\rangle < \cdots < \langle a_n, a_{2n}\rangle < \cdots < \langle a_n\rangle \\
\wedge \\
\langle a_{n+1}\rangle \\
\wedge \\
\vdots
\end{array}$$

Formally.

We define two formulas $\psi(\sigma, Y)$:

$$[(\neg\exists j \in X)\,\big(\sigma(|\sigma| - 1), j\big) \in Y] \qquad \wedge$$
$$\left(R\big(\sigma(|\sigma| - 1)\big) - \bigcup_{\{a:(\exists j \in X)\,(a,j) \in Y\}} R(a) = \{f\big(\sigma(|\sigma| - 1)\big)\} \right)$$

and $\theta(n, Y)$:

$$(\exists \sigma \in T)\Big[\big(\psi(\sigma, Y) \wedge \big((\forall \tau \in T)\,\psi(\tau, Y) \to \sigma \leq_{\mathrm{KB}} \tau\big)\big)$$
$$\wedge \big(n = \sigma(|\sigma| - 1)\big)\Big].$$

ATR_0 contains axioms which guarentee the existence of a set Y such that $H_\theta(\mathrm{KB}(T), Y)$ holds.

We then verify that Y orders all of A, is well founded, and satsifies the desired property. $\square$

Related principles.

STO(F): Let $P = (A, B, R)$ be a matching problem with a unique solution *in which every element has finitely many permissible matches*. Then there is a well-order $(A, <_A)$ such that for every $a \in A$, there is a unique $b \in B$ such that

$$R(a) - R(\{a' : a' <_A a\}) = \{b\}.$$

STO(ω): Let $P = (A, B, R)$ be a matching problem with a unique solution in which every element has finitely many permissible matches. Then there is a well-order $(A, <_A)$ *of type* ω such that for every $a \in A$, there is a unique $b \in B$ such that

$$R(a) - R(\{a' : a' <_A a\}) = \{b\}.$$

Regarding the open reversal.

Theorem (Hughes)
Over RCA_0, ACA_0 *and* $\mathsf{STO}(\omega)$ *are equivalent.*

Theorem (Hughes)
The principle $\mathsf{STO}(\mathsf{F})$ *is provable in* ACA_0.

Theorem (Hughes)
Over RCA_0, $\mathsf{STO}(\mathsf{F})$ *implies* WKL_0.

Future directions.

- Fully classify **STO** and **STO**(**F**) in the reverse mathematical hierarchy.
- Analyze necessary and sufficient conditions for the existence of a solution in the general case.
- Consider matching problems in which R is enumerated.

Thank you for your attention!