Reverse mathematics: an introduction

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Friday, March 30, 2018

S.I.G.M.A. Seminar

A motivating question

"What are the appropriate axioms for mathematics?"

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"What axioms are sufficient and necessary for a given fragment of mathematics?"

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and say $\mathcal{B}$ is sufficient for ξ.

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So set theory with choice is sufficient for Zorn's lemma while set theory without choice is not.

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How do we determine if A was *necessary* to prove ξ and not simply sufficient?

Example:

ZF $\nvdash$ Zorn's lemma   ZF $+$ Axiom of choice $\vdash$ Zorn's lemma

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Note: $\mathcal{B} + A \vdash \xi$ is equivalent to $\mathcal{B} \vdash A \to \xi$.

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Note: $\mathcal{B} + A \vdash \xi$ is equivalent to $\mathcal{B} \vdash A \rightarrow \xi$.

Suppose we could show that the theorem was sufficient to prove the axiom

$$\mathcal{B} \vdash \xi \rightarrow A.$$

This shows that A is necessary to prove ξ as

$$\mathcal{B} \vdash A \leftrightarrow \xi.$$

Relative to $\mathcal{B}$ the axiom A and the theorem ξ are provably equivalent.

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We call this reversing ξ to A and such a proof is called a reversal.

Example:

$$\mathrm{ZF} \vdash \underbrace{\text{Axiom of choice} \to \text{Zorn's lemma}}_{\text{forward for sufficiency}}$$

$$\underbrace{\mathrm{ZF} \vdash \text{ Zorn's lemma } \to \text{ Axiom of choice}}_{\text{reverse for necessity}}$$

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And necessary if we can reverse ξ to A:

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Reverse mathematics is the program of determining which axioms are both sufficient and necessary for proving large fragments of mathematics via this strategy.

Example:

$$\mathsf{ZF} \vdash \text{ Axiom of choice} \leftrightarrow \text{Zorn's lemma}$$

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The goal of this talk is to introduce the resulting 5 axiom systems.

Formal language

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number variables $x, y, z \dots$ and set variables $X, Y, Z, \dots$.

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using logical connectives and quantifiers.

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Logical connectives:

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Distinguished **quantifiers** for each sort of variable:

$$\exists x, \forall y, \exists X, \forall Y$$

Formal language

Example:

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$$\exists X \forall x (x \in X \leftrightarrow \neg (x \in X))$$

is Russel's paradox.

Second order arithmetic

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Is the collection of the following axioms:

- The **basic axioms of arithmetic**
  1. $\forall x \quad \neg(x + 1 = 0)$
  2. $\forall x \forall y \quad x + 1 = y + 1 \rightarrow x = y$
  3. $\forall x \quad x + 0 = x$
  4. $\forall x \forall y \quad x + (y + 1) = (x + y) + 1$
  5. $\forall x \quad x \cdot 0 = 0$
  6. $\forall x \forall y \quad x \cdot (y + 1) = (x \cdot y) + x$
  7. $\forall x \quad \neg(x < 0)$
  8. $\forall x \forall y \quad x < y + 1 \leftrightarrow (x < y \lor x = y)$

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- The **basic axioms of arithmetic.**
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$$\psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x+1)) \rightarrow \forall x \quad \psi(x)$$

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- The **second order comprehension scheme**

$$\exists X \, \forall x \, (x \in X \leftrightarrow \varphi(x)))$$

  where $\varphi(x)$ is any formula of Z_2 in which X does not occur freely.

The base system: RCA_0

The axiom system RCA_0 is the subsystem of Z_2 consisting of the following axioms.

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- The **recursive comprehension scheme**

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \, \forall x \, (x \in X \leftrightarrow \varphi(x)))$$

  where $\varphi(x)$ is any formula with at most one existential quantifier and no other quantifiers and $\psi(x)$ is any formula with at most one universal quantifier and no others.

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Thus, in RCA_0, we do not necessarily have the range of a given function.

RCA_0 is truly a weak axiom system.

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$$\exists Y \forall y (y \in Y \leftrightarrow \exists x (f(x) = y)).$$

Define the compliment of the range with one existential quantifier:

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Membership in Y can be defined via an existential or universal quantifier, so RCA_0 proves that Y exists.

Mathematics in RCA_0

While RCA_0 is a weak axiom system, we can do a modest amount of mathematics. For example,

Theorem

The following are provable in RCA_0*.*

1. *The system* $\mathbb{Z}, +, -, \cdot, 0, 1, <$ *is an ordered integral domain, Euclidean, etc.*
2. *The system* $\mathbb{Q}, +, -, \cdot, 0, 1, <$ *is an ordered field.*
3. *The system* $\mathbb{R}, +, -, \cdot, 0, 1, <, =$ *is an Archimedean ordered field.*
4. *The uncountability of* $\mathbb{R}$*.*
5. *The system* $\mathbb{C}, +, -, \cdot, 0, 1, =$ *is a field.*
6. *The fundamental theorem of algebra.*

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To code finite sequences, we may simply nest this pairing map

$$\begin{aligned}(\ell, m, n) &= (\ell, (m, n)) = (\ell + (m, n))^2 + \ell^2 \\ &= (\ell + (m + n)^2 + m^2)^2 + \ell^2\end{aligned}$$

$$(n_0, n_1, \ldots, n_k) = (n_0, (n_1, \ldots, n_k)).$$

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$$(m, n) +_{\mathbb{Z}} (p, q) = (m + p, n + q)$$

$$(m, n) -_{\mathbb{Z}} (p, q) = (m + q, n + p)$$

$$(m, n) \cdot_{\mathbb{Z}} (p, q) = (m \cdot p + n \cdot q, m \cdot q + n \cdot p)$$

$$(m, n) <_{\mathbb{Z}} (p, q) \leftrightarrow m + q < n + p$$

$$(m, n) =_{\mathbb{Z}} (p, q) \leftrightarrow m + q = n + p$$

Coding the number systems

We then code the rationals $\mathbb{Q}$ via pairs of (codes of) integers (a, b)

$$\begin{aligned} q = \frac{a}{b} &= (a, b) \\ &= ((m_1, n_2), (m_2, n_2)) = ((m_1, n_1) + (m_2, n_2))^2 + (m_1, n_1)^2. \end{aligned}$$

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$$(a, b) +_{\mathbb{Q}} (c, d) = (a \cdot d + b \cdot c, b \cdot d)$$

$$(a, b) -_{\mathbb{Q}} (c, d) = (a \cdot d - b \cdot c, b \cdot d)$$

$$(a, b) \cdot_{\mathbb{Q}} (c, d) = (a \cdot c, b \cdot d)$$

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Now f maps $\mathbb{N}$ to *codes for* $\mathbb{Q}$ so f really maps $\mathbb{N}$ to $\mathbb{N}$.
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We use the usual Cauchy sequence construction of the reals with some technical considerations.Very roughly, a sequence of rationals $x = \langle q_k : k \in \mathbb{N} \rangle$ is a *real number* if

$$\forall k \forall i \, |q_k - q_{k+i}| \leq 2^{-k}.$$

And two real numbers $x = \langle q_k : k \in \mathbb{N} \rangle$ and $y = \langle q'_k : k \in \mathbb{N} \rangle$ equal, written $x = y$, if

$$\forall k \, |q_k - q'_k| \leq 2^{-k+1}.$$

Coding mathematics

We can continue in this way to code

- complete separable metric spaces;
- continuous functions;
- and countable algebraic structures (groups, rings, vector spaces, etc.).

using natural numbers and sets of natural numbers.

This implies that all of the mathematics we see today will really be happening within the natural numbers.

More mathematics in RCA_0

RCA_0 suffices to prove some less trivial facts from countable algebra, real and complex analysis ...

Theorem

The following are provable in RCA_0.

7. *Basics of real linear algebra, including Gaussian Elimination.*
8. *Every countable abelian group has a divisible closure.*
9. *Every countable field has an algebraic closure.*
10. *The intermediate value theorem for continuous real-valued functions: If $f(x)$ is a continuous real-valued function on the unit interval $0 \leq x \leq 1$ and $f(0) < 0 < f(1)$, then there exists c such that $0 < c < 1$ and $f(c) = 0$.*
11. *Every holomorphic function is analytic.*

More mathematics in RCA_0

...the topology of complete separable metric spaces and mathematical logic.

Theorem

The following are provable in RCA_0*.*

12. *The Baire category theorem for complete separable metric spaces : Let* $\langle U_k : k \in \mathbb{N} \rangle$ *be a sequence of dense open sets in* $\widehat{A}$*. Then* $\bigcap_{k \in \mathbb{N}} U_k$ *is dense in* $\widehat{A}$*.*
13. *Urysohn's lemma for complete separable metric spaces : Given (codes for) disjoint closed sets* C_0 *and* C_1 *in* X*, we can effectively find a (code for a) continuous function* $g : X \to [0, 1]$ *such that, for all* $x \in X$ *and* $i \in \{0, 1\}$, $x \in C_i$ *if and only if* $g(x) = i$*.*
14. *The soundness theorem for predicate logic : If* $X \subset \mathrm{SNT}$ *and there exists a countable model* M *such that* $M(\sigma) = 1$ *for all* $\sigma \in X$*, then* X *is consistent.*

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The following are not provable in RCA_0

1. *The Heine/Borel covering lemma: Every covering of the closed interval* $[0, 1]$ *by a sequence of open intervals has a finite subcovering.*
2. *The Bolzano/Weierstraß theorem: Every bounded sequence of real numbers contains a convergent subsequence.*
3. *The perfect set theorem: Every uncountable closed, or analytic, set has a perfect subset.*
4. *The Cantor/Bendixson theorem: Every closed subset of* $\mathbb{R}$, *or of any complete separable metric space, is the union of a countable set and a perfect set.*

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We call such a formula arithmetical.

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The **arithmetical comprehension schema** are the axioms

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The axiom system $\mathsf{ACA}_{\mathbf{0}}$ consists of RCA_0 along with the axioms given in the arithmetical comprehension schema.

Here ACA stands for "arithmetical comprehension axiom."

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Here is an example.

Theorem

Over RCA_0, the following are equivalent

1. ACA_0
2. *For all injective functions $f : \mathbb{N} \to \mathbb{N}$ there exists a set $X \subset \mathbb{N}$ such that X is the range of f.*

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Strategy:

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Then the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(m) = p_2(\pi_Y(m))$.
The definition of Y implies that f is injective.
By item 2, there is a set such that

$$\exists X\,\forall n\,(n \in X \leftrightarrow \exists m(f(m) = n) \leftrightarrow \exists j\,(j,n) \in Y \leftrightarrow \varphi(n))$$

$\square$

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Proof. (Forward direction).
We work in ACA_0 and let G be a countable abelian group.
Via arithmetical comprehension, we can form the set

$$T = \{a \in G : \exists n\,(a^n = 1)\}.$$

It is then straight-forward to show T is a subgroup of G.

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Build G using the generators $x_i, i \in \mathbb{N}$
and the relations $x_{f(m)}^{(2m+1)} = 1$ for all $m \in \mathbb{N}$.

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G is the set of finite formal products $\Pi x_i^{n_i}$ where $n_i \in \mathbb{Z}$ and

$$\forall m \, (m < |n_i| \to f(m) \neq i).$$

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As we only need a bounded quantifier, G exists by recursive comprehension.

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So X is the range of f.

By the previous theorem, Item 2 implies arithmetical comprehension and the reversal is complete.

$\square$

Countable algebra and ACA_0

Theorem

Over RCA_0*, the following are equivalent*

1. ACA_0
2. *Every countable Abelian group has a unique divisible closure.*
3. *Every countable commutative ring has a maximal ideal.*
4. *Every countable vector space over a countable field has a basis.*
5. *Every countable field (of characteristic 0) has a transcendence basis.*

Analysis and ACA_0

Theorem

Over RCA_0*, the following are equivalent*

1. ACA_0
6. *Every Cauchy sequence of real numbers is convergent.*
7. *The Bolzano/Weierstraß theorem: Every bounded sequence of real numbers contains a convergent subsequence.*
8. *The Ascoli lemma: Every bounded equicontinuous sequence of real=valued continuous functions on a bounded interval has a uniformly convergent subsequence.*

A few more results and ACA_0

Theorem

Over RCA_0*, the following are equivalent*

1. ACA_0

9. *König's lemma: Every infinite, finitely branching tree has an infinite path.*

10. *Ramsey's theorem for colorings of* $[\mathbb{N}]^k$*,* $k > 2$*: For all finite colorings of increasing sequences of length* k *of* $\mathbb{N}$*, there is an infinite subset* $X \subset \mathbb{N}$ *such that* $[X]^k$ *is homogeneous in color.*

$\Pi^1_1-\mathsf{CA}_0$

Π^1_1–CA_0

Definition

The Π^1_1 **comprehension schema** are the axioms

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where φ is any formula of the form $\forall Y\,\theta$ where θ has no set quantifiers.

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Definition

The axiom systems $\Pi^1_1{-}\mathrm{CA}_0$ consists of RCA_0 along with the axioms given in the Π^1_1 comprehension schema.

The reverse mathematics of $\Pi^1_1-\mathsf{CA}_0$

Theorem

Over RCA_0, the following are equivalent:

1. $\Pi^1_1-\mathsf{CA}_0$
2. *Every countable Abelian group is the direct sum of a divisible group and a reduced group.*
3. *The Cantor/Bendixson theorem: Every closed subset of $\mathbb{R}$, or of any complete separable metric space, is the union of a countable set and a perfect set.*
4. *Silver's theorem: For every Borel equivalence relation with uncountably many equivalence classes, there exists a nonempty perfect set of inequivalent elements.*
5. *Every tree has a largest perfect subtree.*
6. *Every G_δ set in $[\mathbb{N}]^\mathbb{N}$ has the Ramsey property.*

Weak König's Lemma and WKL_0

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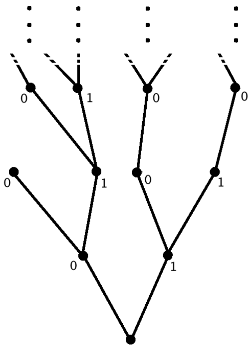

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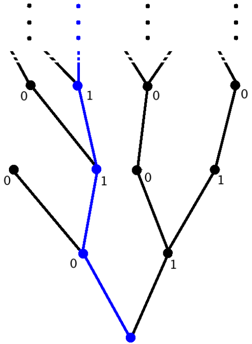

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Definition

The axiom system WKL_0 consists of the axioms of RCA_0 along with weak König's lemma.

The reverse mathematics of WKL_0

Theorem

Over RCA_0, the following are equivalent:

1. WKL_0
2. *The Heine/Borel covering lemma: Every covering of the closed interval $[0, 1]$ by a sequence of open intervals has a finite subcovering.*
3. *The maximum principle: Every continuous real-valued function on $[0, 1]$ attains a supremum.*
4. *Every continuous real-valued function on $[0, 1]$ is Riemann integrable.*

The reverse mathematics of WKL_0

Theorem

Over RCA_0, the following are equivalent:

1. WKL_0
5. *Cauchy's integral theorem: If f is holomorphic on an open set $D \subset \mathbb{C}$, and γ is a triangular path in D, then*

$$\int_\gamma f(z)\,dz = 0$$

6. *The local existence theorem for solutions of ordinary differential equations.*
7. *Brouwer's fixed point theorem: Every uniformly continuous function $\phi : [0,1]^n \to [0,1]^n$ has a fixed point.*

The reverse mathematics of WKL_0

Theorem

Over RCA_0, the following are equivalent:

1. WKL_0

8. *The separable Hahn/Banach theorem: If f is a bounded linear functional on a subspace of a separable Banach space, and if $||f|| \leq 1$, then f has an extension $\hat{f}$ to the whole space such that $||\hat{f}|| \leq 1$.*

9. *Every countable commutative ring has a prime ideal.*

10. *Every countable field (of characteristic 0) has a unique algebraic closure.*

11. *Gödel's completeness theorem: Every countable set of sentences in the predicate calculus has a countable model.*

WKL_0 and ACA_0

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Clearly, by increasing set comprehension

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Definition

The axiom system ATR_0 consists of the axioms of RCA_0 along with axioms for arithmetical transfinite recursion.

The reverse mathematics of ATR_0

Theorem

Over RCA_0, the following are equivalent:

1. ATR_0
2. *Any two countable well orderings are comparable.*
3. *The perfect set theorem: Every uncountable closed, or analytic, set has a perfect subset.*
4. *Lusin's separation theorem: Any two disjoint analytic sets can be separated by a Borel set.*
5. *The domain of any single-valued Borel relation is Borel.*
6. *Ulm's theorem: Any two countable reduced Abelian p-groups which have the same Ulm invariants are isomorphic.*
7. *The open Ramsey theorem: Every open subset of $[\mathbb{N}]^{\mathbb{N}}$ has the Ramsey property.*

The big five

We now have seen five subsystems of second order arithmetic which serve as appropriate axiomatizations of substantial portions of mathematics.

These systems are known as *the big five*:

$$\mathsf{RCA}_0 \quad \mathsf{WKL}_0 \quad \mathsf{ACA}_0 \quad \mathsf{ATR}_0 \quad \Pi^1_1{-}\mathsf{CA}_0$$

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Though we have shown many theorems equivalent to one of these, many more theorems have been shown to fit nicely into this hierarchy in the 40+ years since their introduction.

Because of this, we consider reverse mathematics to be an important partial answer to the motivating question

what are the appropriate axioms of reverse mathematics?

The big five

$$\begin{array}{ccl}
\Pi^1_1\text{–}\mathsf{CA}_0 & \iff & \textit{The Cantor/Bendixson theorem} \\
\Downarrow & & \\
\mathsf{ATR}_0 & \iff & \textit{The perfect set theorem} \\
\Downarrow & & \\
\mathsf{ACA}_0 & \iff & \textit{The Bolzano/Weierstraß theorem} \\
\Downarrow & & \\
\mathsf{WKL}_0 & \iff & \textit{The Heine/Borel covering lemma} \\
\Downarrow & & \\
\mathsf{RCA}_0 & \iff & \textit{The intermediate value theorem}
\end{array}$$

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Thank you!