Reverse mathematics and marriage problems with finitely many solutions

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Reverse Mathematics

Goal: To determine the exact set existence axioms needed to prove a familiar theorem.

Method: Prove results of the form $\mathrm{RCA}_0 \vdash \mathbf{AX} \leftrightarrow \mathbf{THM}$

The base system RCA_0:

Second order arithmetic: integers n and sets of integers X

Induction scheme: restricted to Σ^1_0 formulas

$$(\psi(0) \wedge \forall n(\psi(n) \to \psi(n+1))) \to \forall n \psi(n)$$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension:

If $\theta \in \Sigma^1_0$ and $\psi \in \Pi^1_0$, and $\forall n(\theta(n) \leftrightarrow \psi(n))$, then there is a set X such that $\forall n(n \in X \leftrightarrow \theta(n))$.

More set comprehension axioms

Weak König's Lemma: (WKL_0) If T is an infinite tree in which each node is labeled 0 or 1, then T contains an infinite path.

Arithmetical comprehension: (ACA_0) If $\theta(n)$ does not have any set quantifiers, then there is an X such that $\forall n (n \in X \leftrightarrow \theta(n))$.

Theorem (Friedman)

RCA_0 *proves that the following are equivalent:*

1. ACA_0
2. (KL) König's Lemma: *If T is an infinite tree and every level of T is finite, then T contains an infinite path.*

Marriage problems

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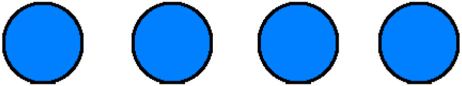

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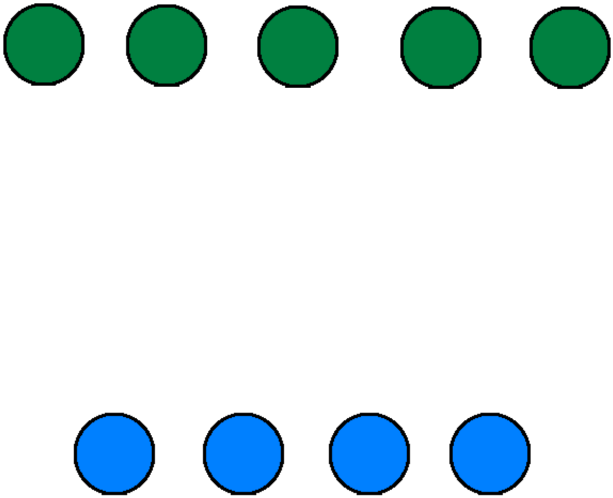

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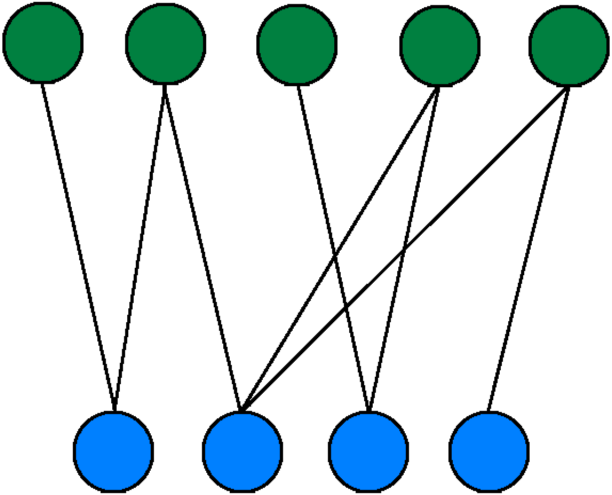

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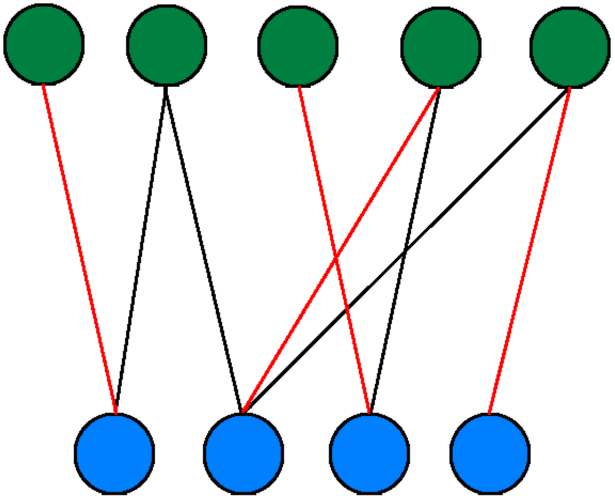

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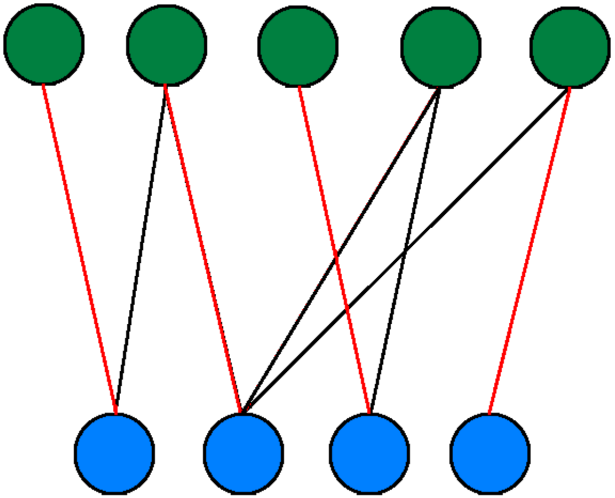

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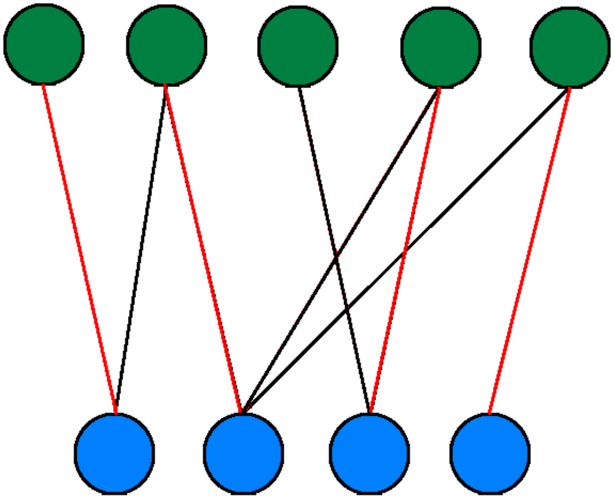

Notation

A marriage problem M consists of three sets: B, G and R

B is the set of boys, G is the set of girls, and $R \subseteq B \times G$

where $(b, g) \in R$ implies that "boy b knows girl g"

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$$G(b) = \{g \in G \mid (b, g) \in R\}.$$

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Assume $G(b)$ to be finite for all $b \in B$.

Match makers

A *solution* to $M = (B, G, R)$ is an injection

$$f : B \to G$$

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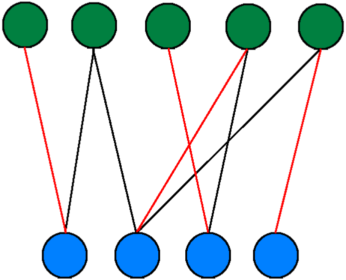

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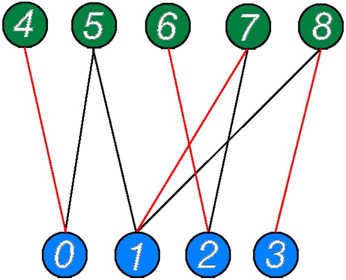

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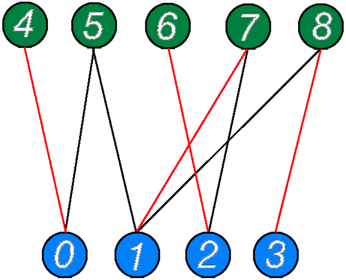

$$B = \{0, 1, 2, 3\}$$
$$G = \{4, 5, 6, 7, 8\}$$

$$f = \begin{cases} 0 \to 4 \\ 1 \to 7 \\ 2 \to 6 \\ 3 \to 8 \end{cases}$$

f is a solution.

When does a marriage problem have...

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Theorem (Hall)

A marriage problem $M = (B, G, R)$ *has a solution if and only if* $|G(B_0)| \geq |B_0|$ *for every* $B_0 \subset B$.

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a unique solution?

Theorem (Hirst, Hughes)

A marriage problem $M = (B, G, R)$ has a unique solution if and only if there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that for every $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$.

Marriage problems with k many solutions

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Theorem (Hirst, Hughes)

If a marriage problem $M = (B, G, R)$ has exactly k solutions, $f_1, \ldots, f_k$, then there is a finite set $B_0 \subset B$ such that for all $i < j \leq k$ and $b \in B$, if $f_i(b) \neq f_j(b)$ then $b \in B_0$.

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Theorem (Hirst, Hughes)

A marriage problem $M = (B, G, R)$ has exactly k solutions if and only if there is some finite set of boys such that M restricted to this set has exactly k solutions and each solution extends uniquely to a solution of M.

Ordering marriage problems with k many solutions

Theorem (Hirst, Hughes)

Suppose $M = (B, G, R)$ is a marriage problem with exactly k solutions: $f_1, f_2, \ldots, f_k$. Then there is a finite set $F \subseteq B$ and a sequence of k sequences $\langle b^i_j \rangle_{j \geq 1}$ for $1 \leq i \leq k$ such that the following hold:

(i) *M restricted to F has exactly k solutions, each corresponding to f_i restricted to F for some i.*

(ii) *For each $1 \leq i \leq k$, the sequence $\langle b^i_j \rangle_{j \geq 1}$ enumerates all the boys not included in F.*

(iii) *For each $1 \leq i \leq k$ and each $n \in \mathbb{N}$,*

$$|G(\{b^i_1, b^i_2, \ldots, b^i_n\}) - f_i(F)| = n$$

Reverse mathematics and marriage theorems

Often, the strength of the marriage theorems we've considered depend upon whether the underlying marriage problem is finite, bounded or infinite.

Theorem

Over RCA_0*, the following are equivalent:*

1. ACA_0
2. (Hirst.) *An infinite marriage problem* $M = (B, G, R)$ *has a solution only if* $|G(B_0)| \geq |B_0|$ *for every* $B_0 \subset B$*.*
3. (Hirst, Hughes.) *An infinite marriage problem* $M = (B, G, R)$ *has a unique solution only if there is an enumeration of the boys* $\langle b_i \rangle_{i \geq 1}$ *such that for every* $n \geq 1$*,* $|G(\{b_1, b_2, \ldots, b_n\})| = n$*.*
4. (Hirst, Hughes.) *An infinite marriage problem* $M = (B, G, R)$ *has exactly* k *solutions only if there is some finite set of boys such that* M *restricted to this set has exactly* k *solutions and each solution extends uniquely to a solution of* M*.*

What to prove, what to prove?

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Equivalently:

$$\mathsf{RCA}_0 \vdash (\mathsf{ACA}_0 \Rightarrow \text{Item 2}) \wedge (\text{Item 2} \Rightarrow \mathsf{ACA}_0).$$

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A (sketch of a) reversal

Recall ACA_0 is equivalent to König's lemma.

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$$((\text{Item 2} \Rightarrow \mathsf{KL}) \wedge (\mathsf{KL} \iff \mathsf{ACA}_0)) \\ \implies (\text{Item 2} \Rightarrow \mathsf{ACA}_0)$$

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Goal:

Use Item 2 to prove König's lemma.

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$$((\text{Item 2} \Rightarrow \text{KL}) \wedge (\text{KL} \iff \mathsf{ACA}_0)) \\ \implies (\text{Item 2} \Rightarrow \mathsf{ACA}_0)$$

Goal:

Use Item 2 to prove König's lemma.

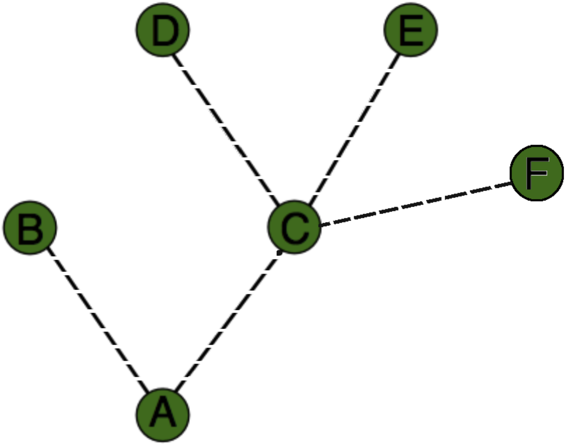
The contrapositive of König's lemma will be easier to prove.

Theorem

If T is a tree with no infinite paths and every level of T is finite, then T is a finite tree.

A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls.



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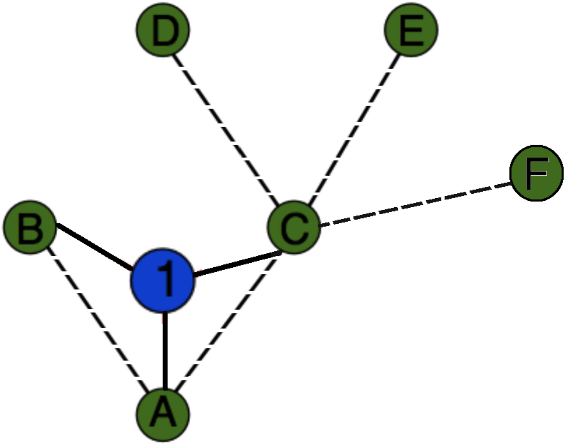
Here's a tree with no infinite paths. Nodes are girls. Add a boy.



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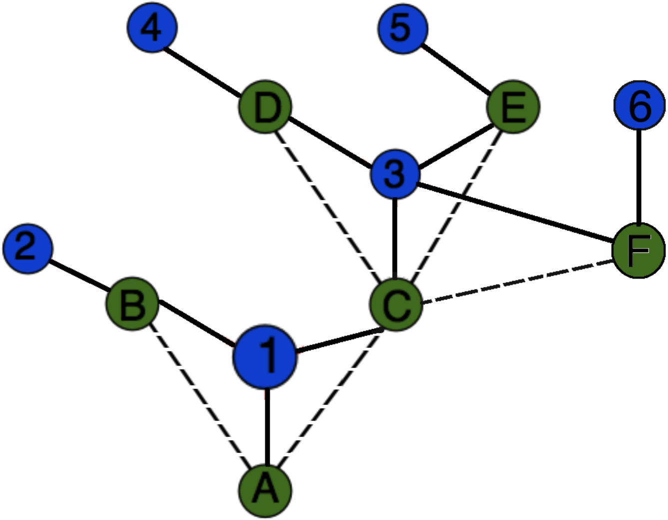
Here's a tree with no infinite paths. Nodes are girls. Complete the society.



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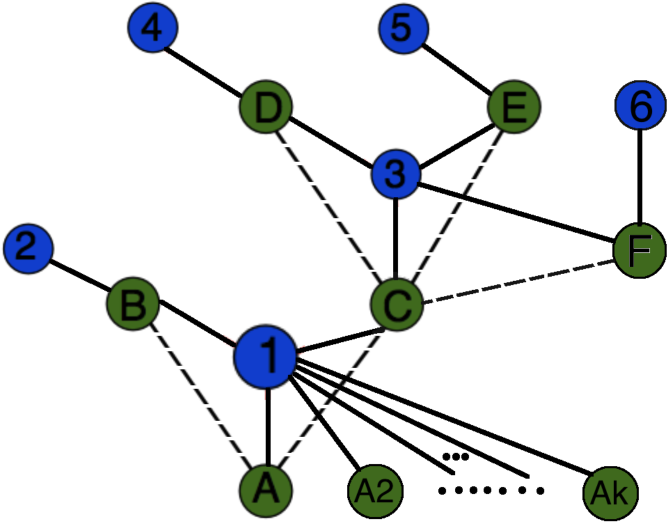
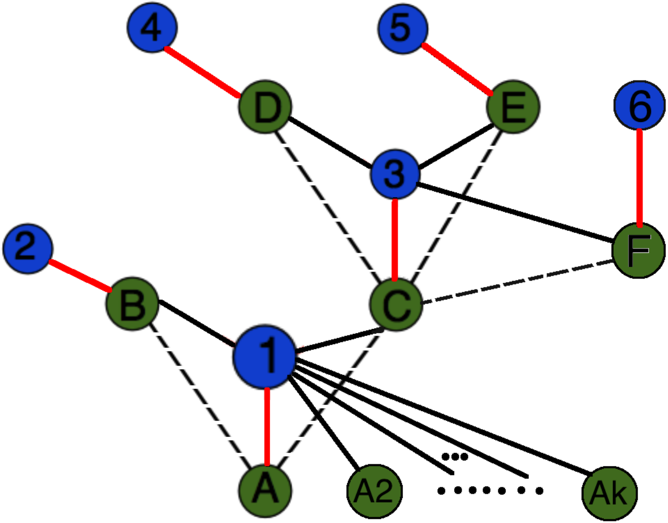
Here's a tree with no infinite paths. Nodes are girls. Complete the society. Add $k-1$ girls to the first boy.



A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls. Complete the society. Add $k-1$ girls to the first boy. There are k solutions.



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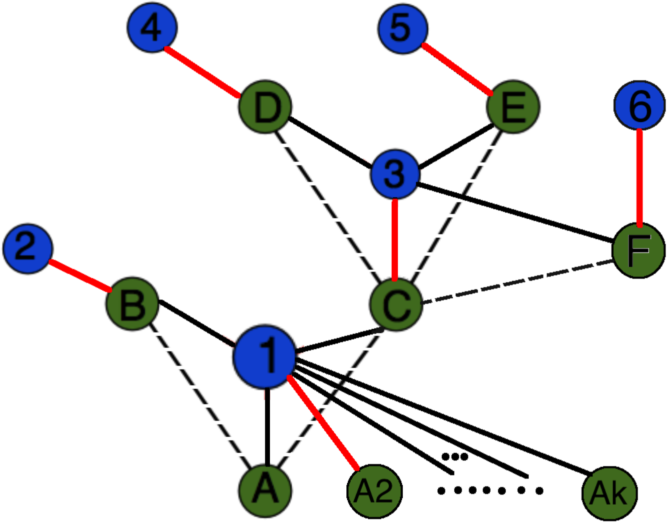

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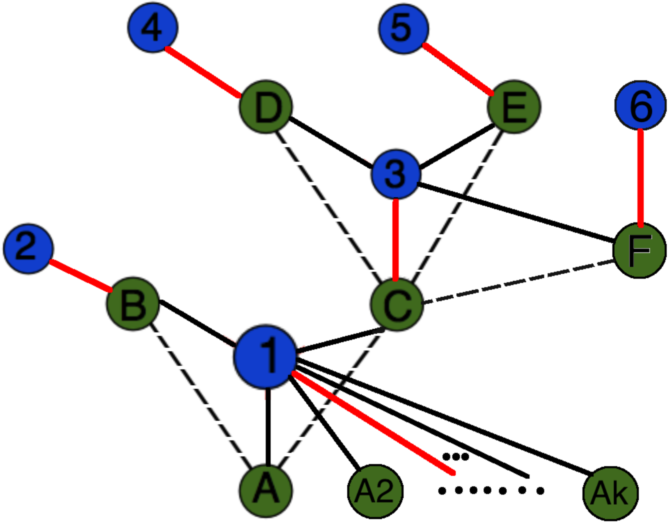

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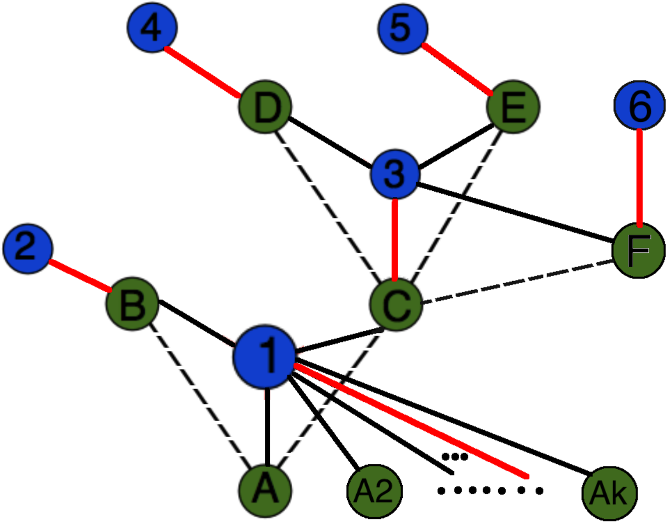

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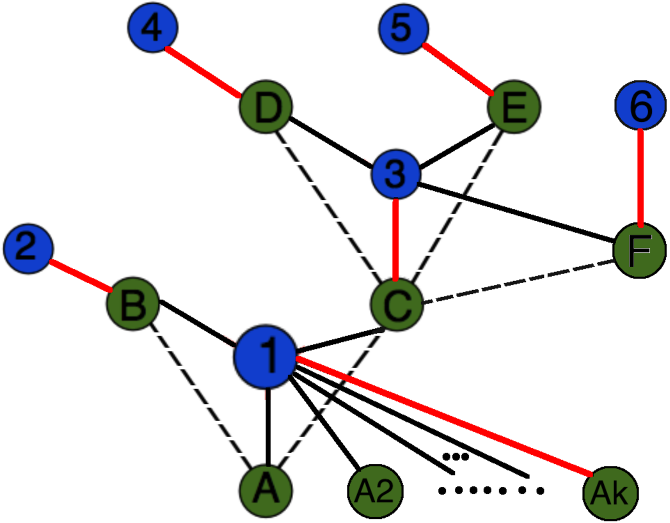
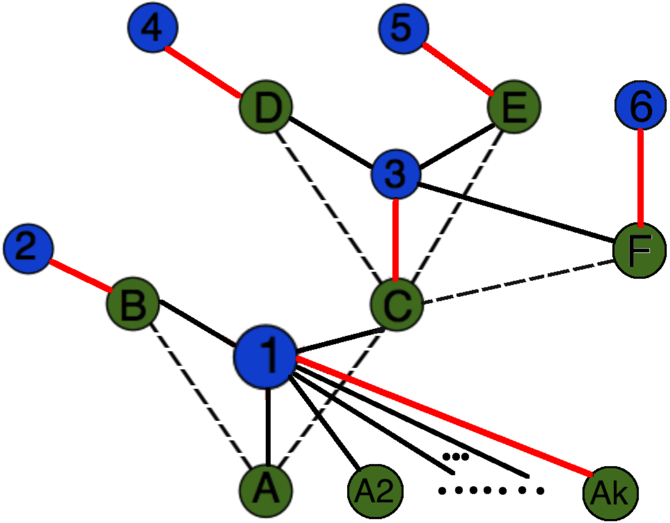


A (sketch of a) reversal

Here's a tree with no infinite paths. There are k solutions. By Item 2, the finite set F exists. Boy 1 and any successor of Boy 1 must be in F. The tree is finite.



Reverse mathematics and marriage problems (cont.)

Theorem

(RCA_0) *If $M = (B, G, R)$ is a marriage problem with a unique solution, then some boy knows exactly one girl.*

Theorem

Over RCA_0, the following are equivalent:

1. WKL_0
2. *If a marriage problem $M = (B, G, R)$ has exactly k solutions, $f_1, \ldots, f_k$, then there is a finite set $B_0 \subset B$ such that for all $i < j \leq k$ and $b \in B$, if $f_i(b) \neq f_j(b)$ then $b \in B_0$.*

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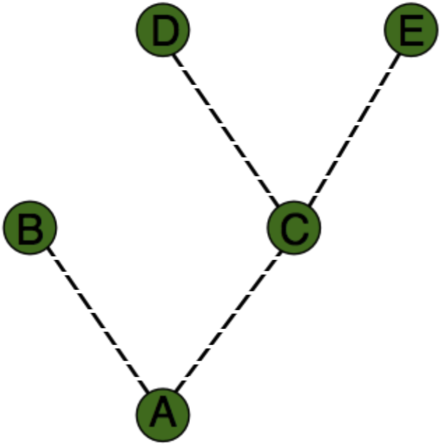
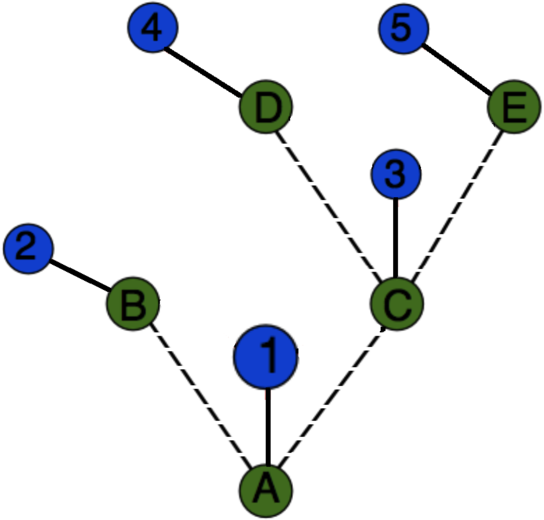


A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls. Add boys.



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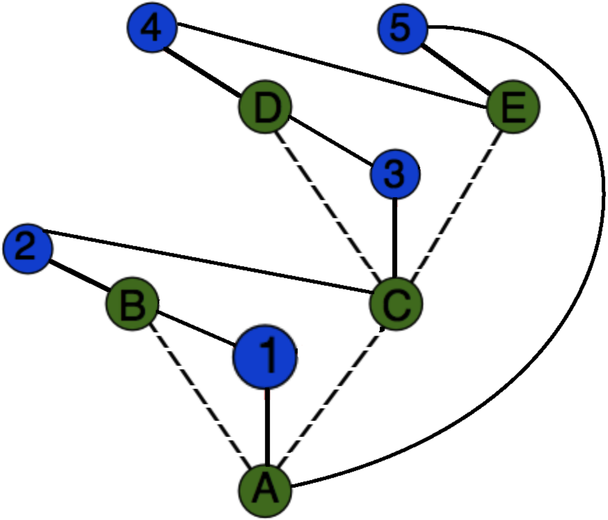
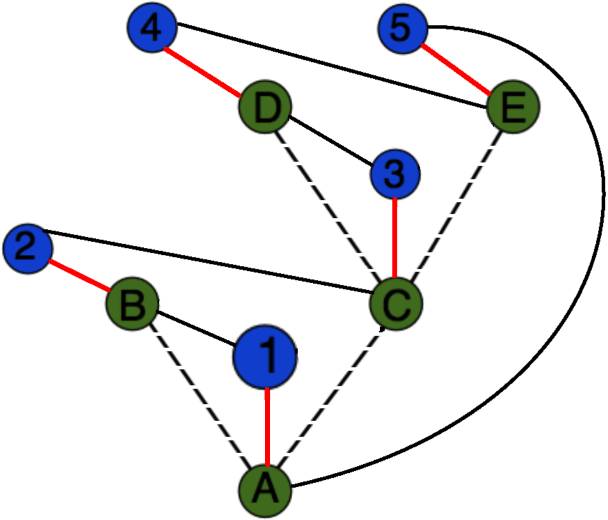


A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls. Complete the society. There are exactly two solutions which differ at every boy.



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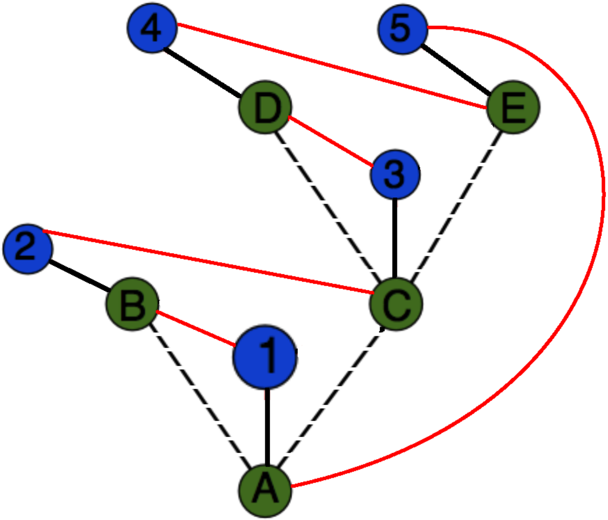
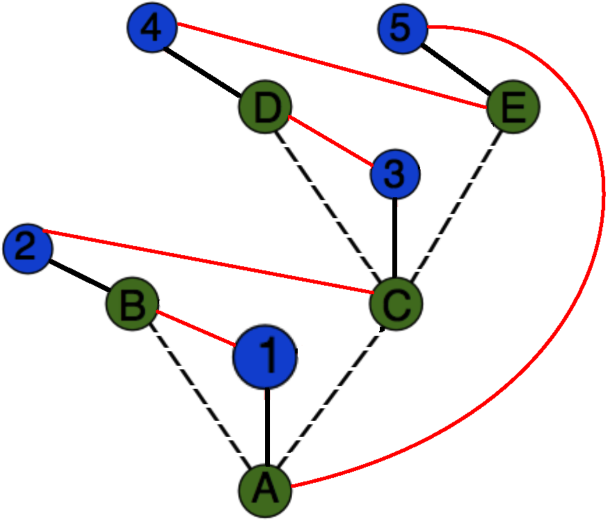


A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls. Complete the society. There are exactly two solutions which differ at every boy. By Item 2, all boys are in a finite set. The tree is finite.



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Thank you!