Reverse mathematics and marriage problems: a few new results

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Marriage problems

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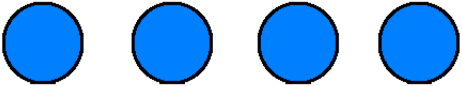

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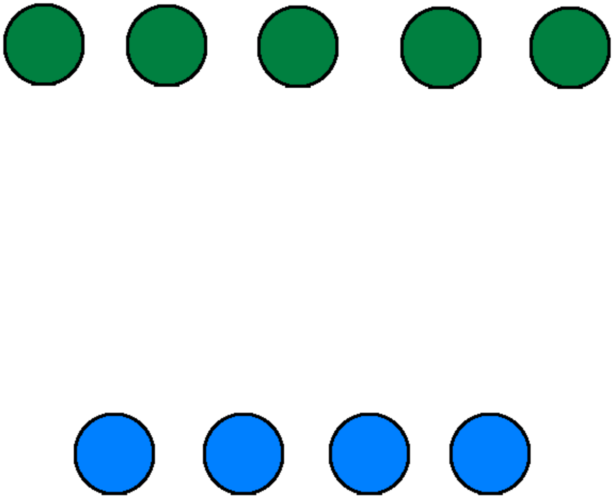

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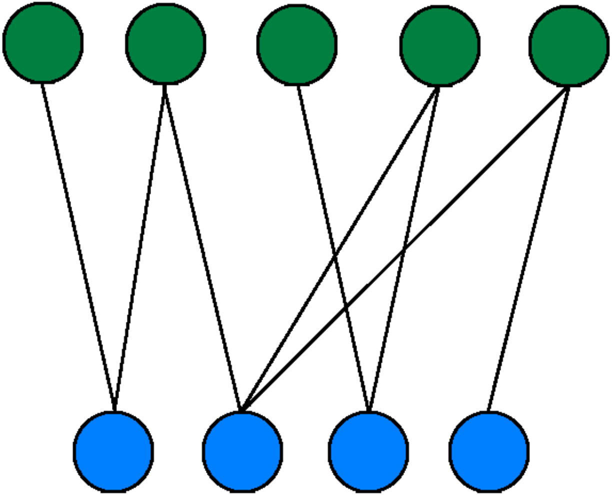

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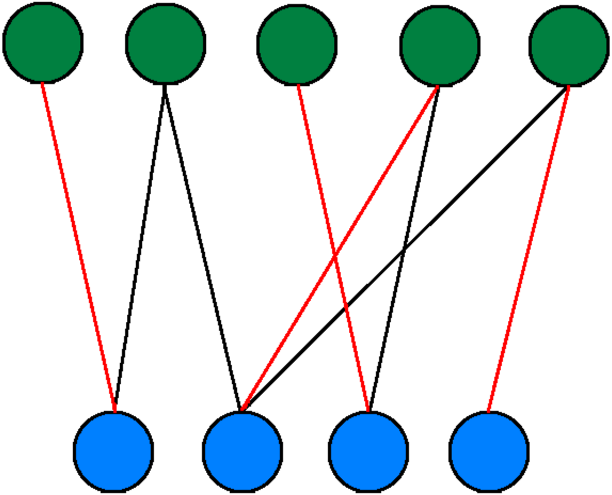

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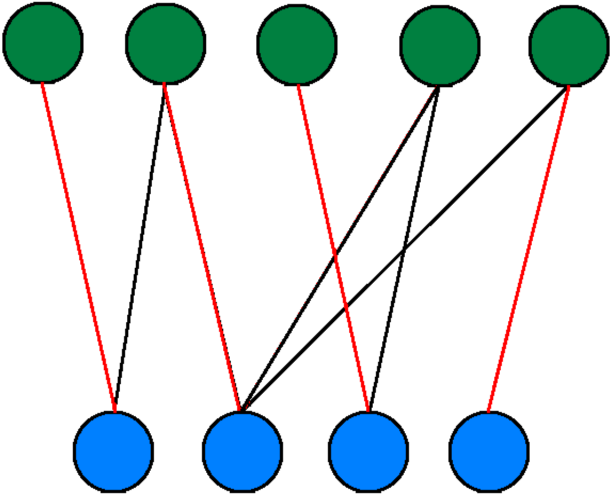

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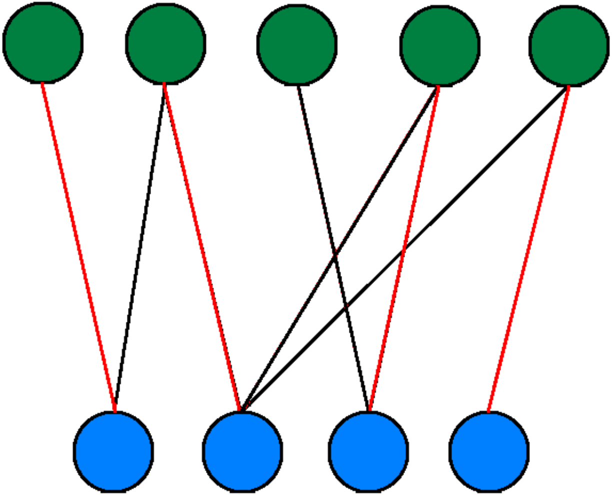

Notation

A marriage problem M consists of three sets: B, G and R

B is the set of boys, G is the set of girls, and $R \subseteq B \times G$

where $(b, g) \in R$ implies that "boy b knows girl g"

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$G(b)$ is *not* a function.

Assume $G(b)$ to be finite for all $b \in B$.

Match makers

A *solution* to $M = (B, G, R)$ is an injection

$$f : B \to G$$

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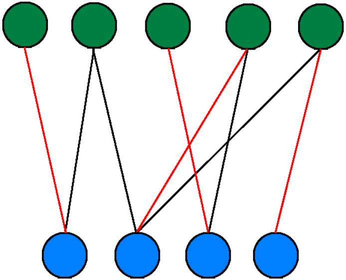

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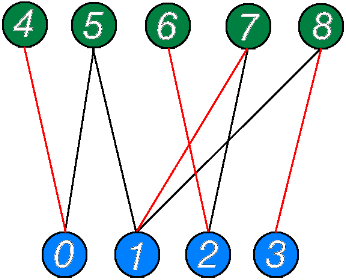

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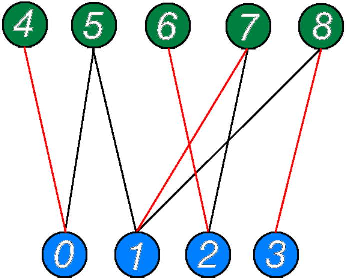

$B = \{0, 1, 2, 3\}$
$G = \{4, 5, 6, 7, 8\}$

$$f = \begin{cases} 0 \to 4 \\ 1 \to 7 \\ 2 \to 6 \\ 3 \to 8 \end{cases}$$

f is a solution.

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Theorem (Hall)

A marriage problem $M = (B, G, R)$ has a solution if and only if $|G(B_0)| \geq |B_0|$ for every $B_0 \subset B$.

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Theorem (Hirst, Hughes)

A marriage problem $M = (B, G, R)$ has a unique solution if and only if there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that for every $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$.

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Theorem (Hirst, Hughes)

A marriage problem $M = (B, G, R)$ has exactly k solutions if and only if there is some finite set of boys such that M restricted to this set has exactly k solutions and each solution extends uniquely to a solution of M.

Ordering a marriage problem with k many solutions

Theorem (Hirst, Hughes)

Suppose $M = (B, G, R)$ is a marriage problem with exactly k solutions: $f_1, f_2, \ldots, f_k$. Then there is a finite set $F \subseteq B$ and a sequence of k sequences $\langle b^i_j \rangle_{j\geq 1}$ for $1 \leq i \leq k$ such that the following hold:

(i) *M restricted to F has exactly k solutions, each corresponding to f_i restricted to F for some i.*

(ii) *For each $1 \leq i \leq k$, the sequence $\langle b^i_j \rangle_{j\geq 1}$ enumerates all the boys not included in F.*

(iii) *For each $1 \leq i \leq k$ and each $n \in \mathbb{N}$,*

$$|G(\{b^i_1, b^i_2, \ldots, b^i_n\}) - f_i(F)| = n$$

Reverse mathematics

An introduction

Reverse mathematics is motivated by a foundational question:

Question: Exactly which axioms do we really need to prove a given theorem?

The program of reverse mathematics seeks to prove results of the form:

> Over a weak base theory B, axiom A is equivalent to theorem T.

This naturally leads to the idea of the strength of a theorem.

To sharpen this notion of strength, we restrict our attention to set existence axioms.

I.e., the more complex sets axiom A asserts the existence of, the stronger the theorem T.

A weak base theory

We take RCA_0 as our weak base theory:

axioms for arithmetic;
limited induction;
comprehension for computable sets.

RCA stands for "recursive comprehension axiom"

(recursive $\sim$ computable)

RCA_0 proves the *intermediate value theorem* and the *uncountability of* $\mathbb{R}$.

RCA_0 does **not** prove the *existence of Riemann integrals.*

Another set comprehension axiom

ACA_0 adds comprehension for arithmetical sets.

This adds an immense amount of set comprehension, e.g., the existence of many noncomputable sets.

ACA_0 is strong enough to prove the *Bolzano-Weierstraß theorem* and that *every countable vector space over $\mathbb{Q}$ has a basis.*

Theorem (Friedman)

Over RCA_0*, the following are equivalent:*

1. ACA_0
2. (KL) König's Lemma: *If T is an infinite tree and every level of T is finite, then T contains an infinite path.*

Reverse mathematics and marriage theorems

In general, the strength of the marriage theorems we've considered depend upon whether the underlying marriage problem is finite or infinite.

Theorem

The following are provable in RCA_0

1. (Hirst.) *A finite marriage problem* $M = (B, G, R)$ *has a solution only if* $|G(B_0)| \geq |B_0|$ *for every* $B_0 \subset B$.
2. (Hirst, Hughes.) *A finite marriage problem* $M = (B, G, R)$ *has a unique solution only if there is an enumeration of the boys* $\langle b_i \rangle_{i \geq 1}$ *such that for every* $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$.
3. (Hirst, Hughes.) *A finite marriage problem* $M = (B, G, R)$ *has exactly* k *solutions only if there is some finite set of boys such that* M *restricted to this set has exactly* k *solutions and each solution extends uniquely to a solution of* M.

Reverse mathematics and marriage theorems

If the underlying marriage problem is infinite, the marriage theorem becomes much stronger:

Theorem

Over RCA_0, the following are equivalent:

1. ACA_0
2. (Hirst.) *An infinite marriage problem $M = (B, G, R)$ has a solution only if $|G(B_0)| \geq |B_0|$ for every $B_0 \subset B$.*
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4. (Hirst, Hughes.) *An infinite marriage problem $M = (B, G, R)$ has exactly k solutions only if there is some finite set of boys such that M restricted to this set has exactly k solutions and each solution extends uniquely to a solution of M.*

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Equivalently:

$$\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \iff \text{Item 2.}$$
$$(\mathsf{ACA}_0 \Rightarrow \text{Item 2}) \wedge (\text{Item 2} \Rightarrow \mathsf{ACA}_0).$$

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A (sketch of a) reversal

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Goal:

Use Item 2 to prove König's lemma.

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Goal:

Use Item 2 to prove König's lemma.

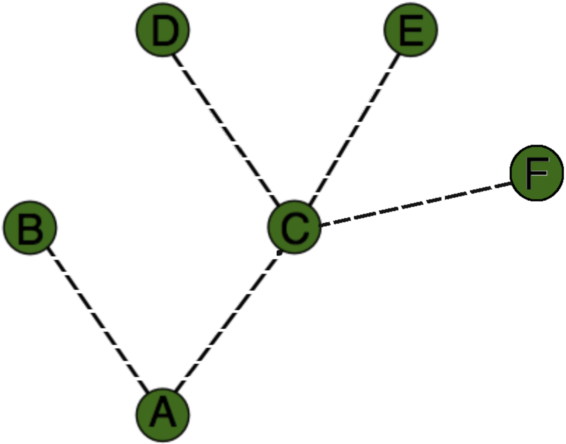
The contrapositive of König's lemma will be easier to prove.

Theorem

If T is a tree with no infinite paths and every level of T is finite, then T is a finite tree.

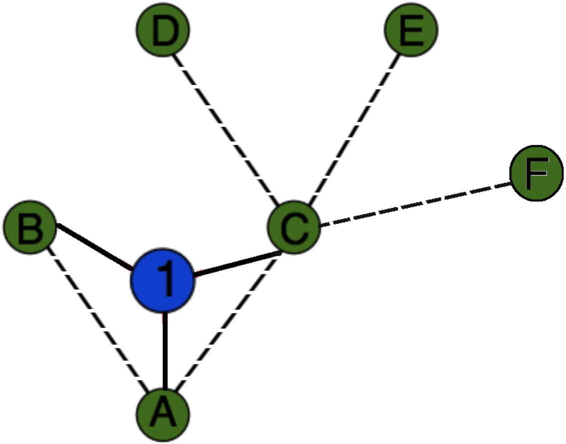
A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls.



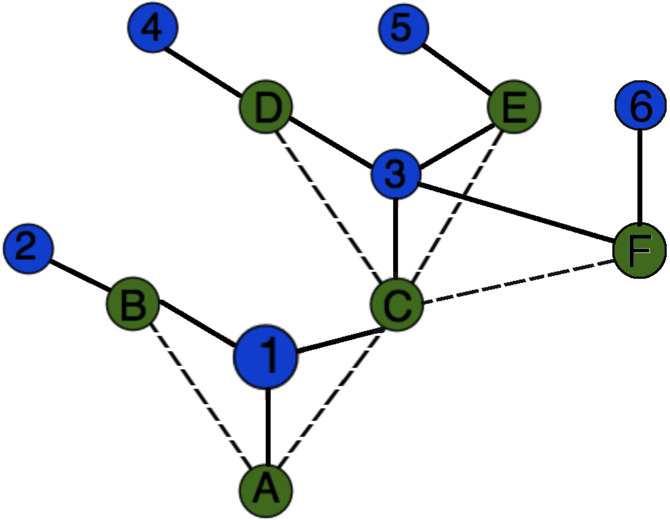
A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls. Add a boy.



A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls. Complete the society.



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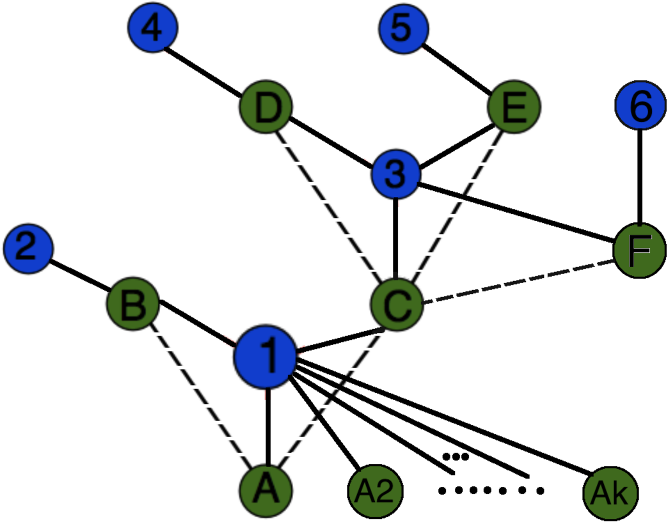
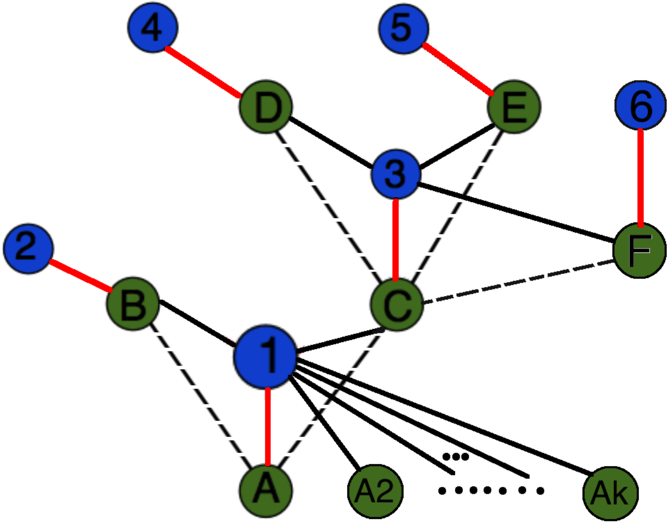
Here's a tree with no infinite paths. Nodes are girls. Complete the society. Add $k-1$ girls to the first boy.



A (sketch of a) reversal

Here's a tree with no infinite paths. Nodes are girls. Complete the society. Add $k-1$ girls to the first boy. There are k solutions.



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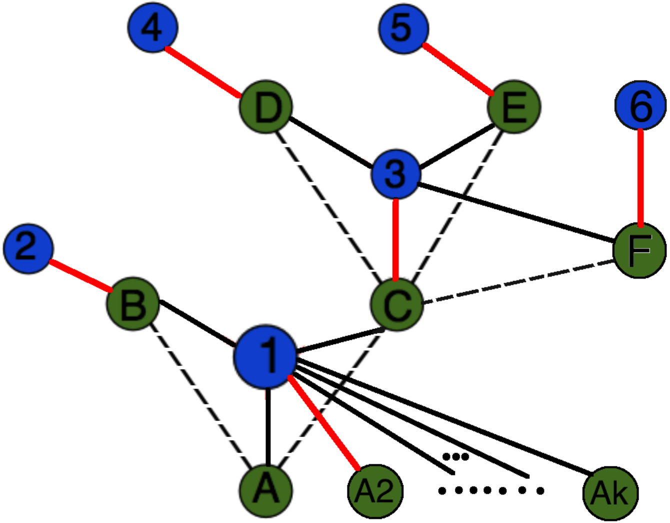

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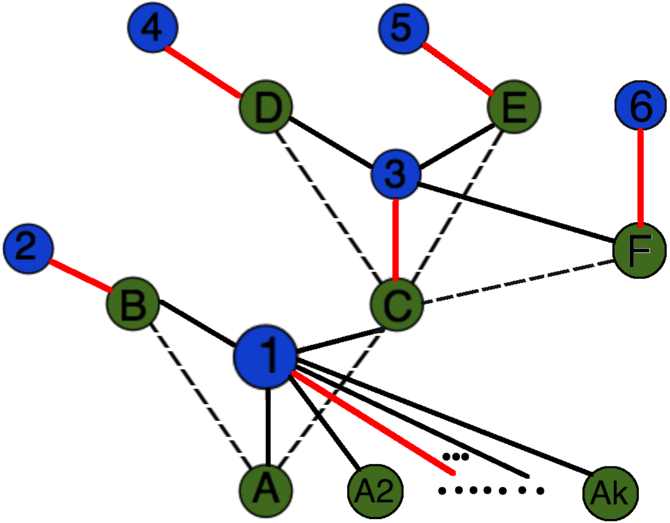

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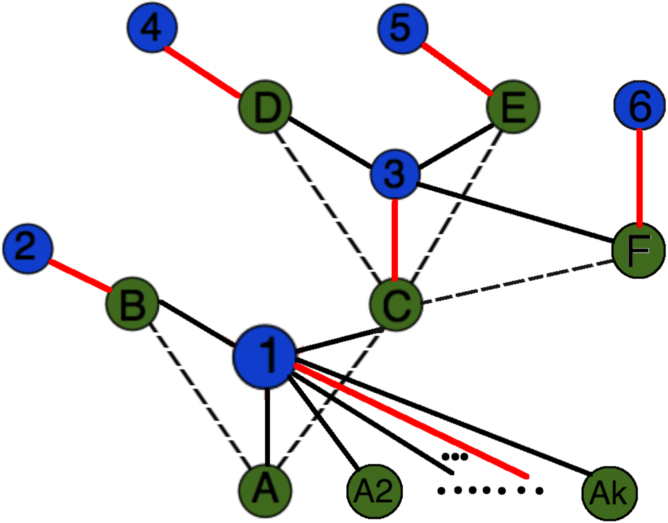

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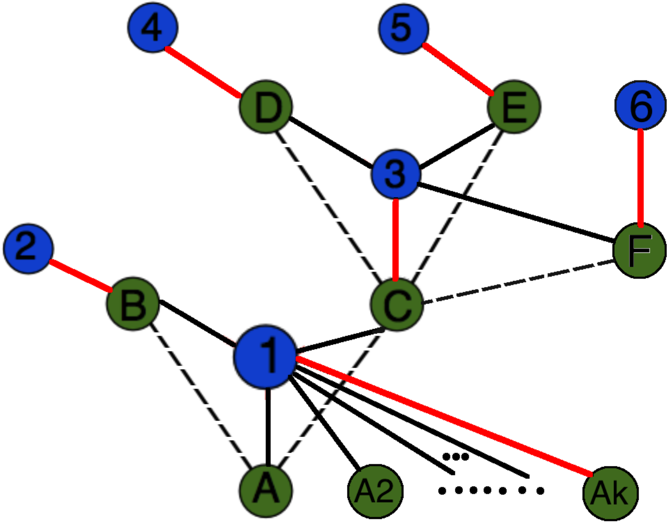
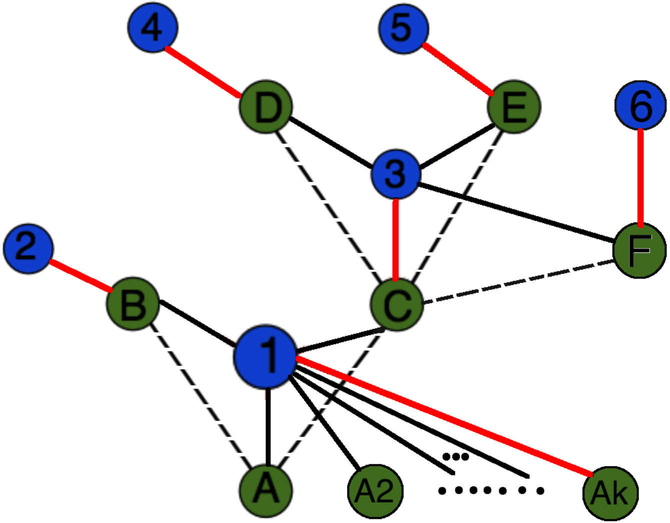


A (sketch of a) reversal

Here's a tree with no infinite paths. There are k solutions. By Item 2, the finite set F exists. Boy 1 and any successor of Boy 1 must be in F. The tree is finite.



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Thank you!