Reverse Mathematics and Marriage Problems

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The University of North Carolina Greensboro

Agenda

- I: Marriage Problems
- I: Previous Results
- I: New Results
- II: Reverse Mathematics

I: Marriage Problems

Marriage Problems

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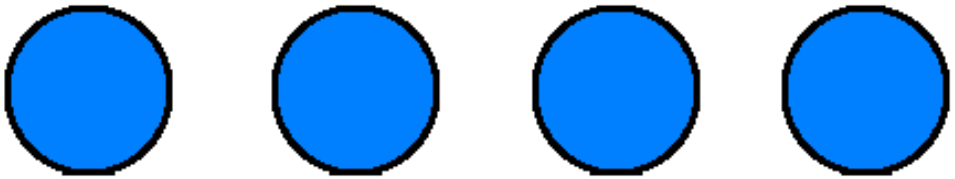

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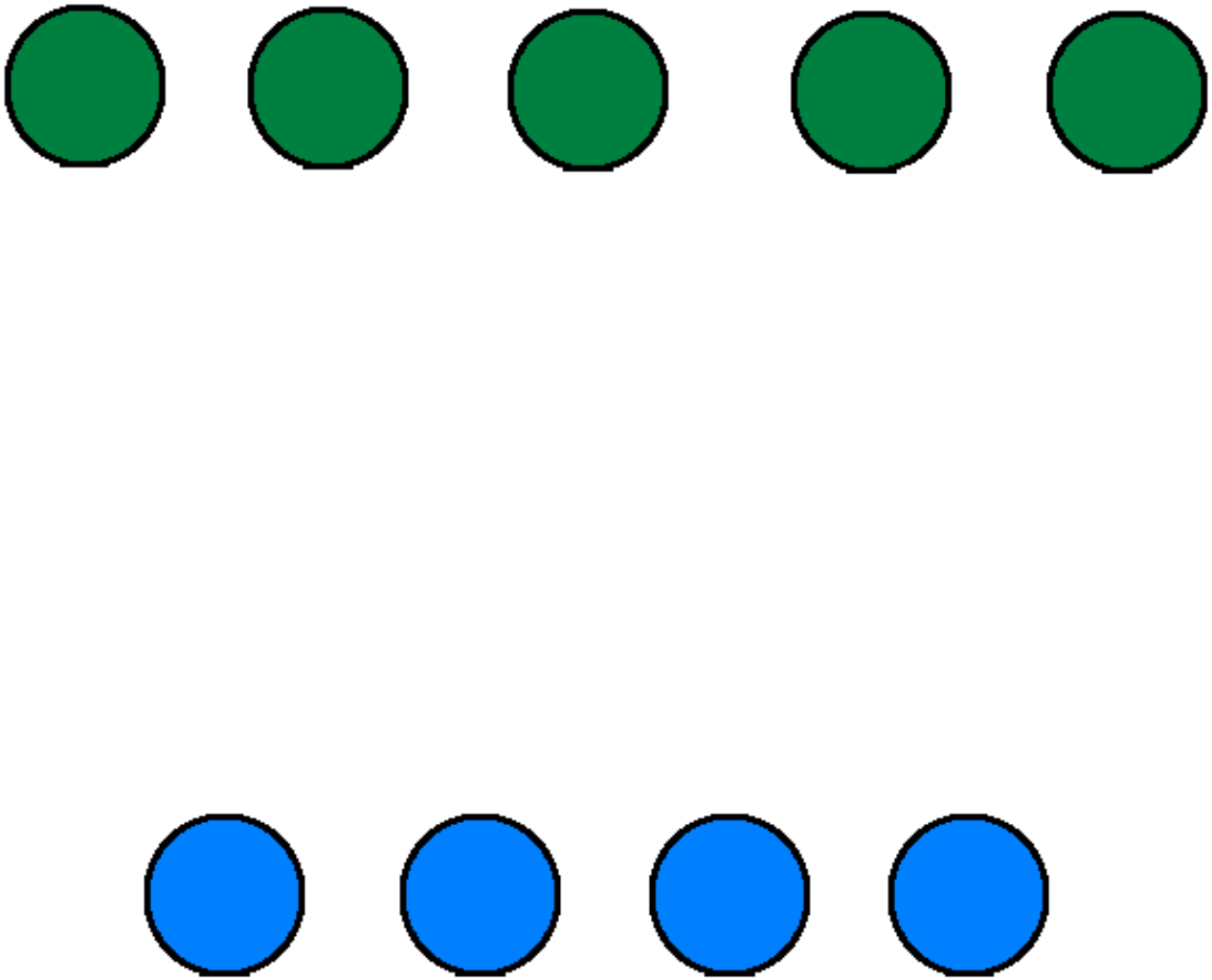

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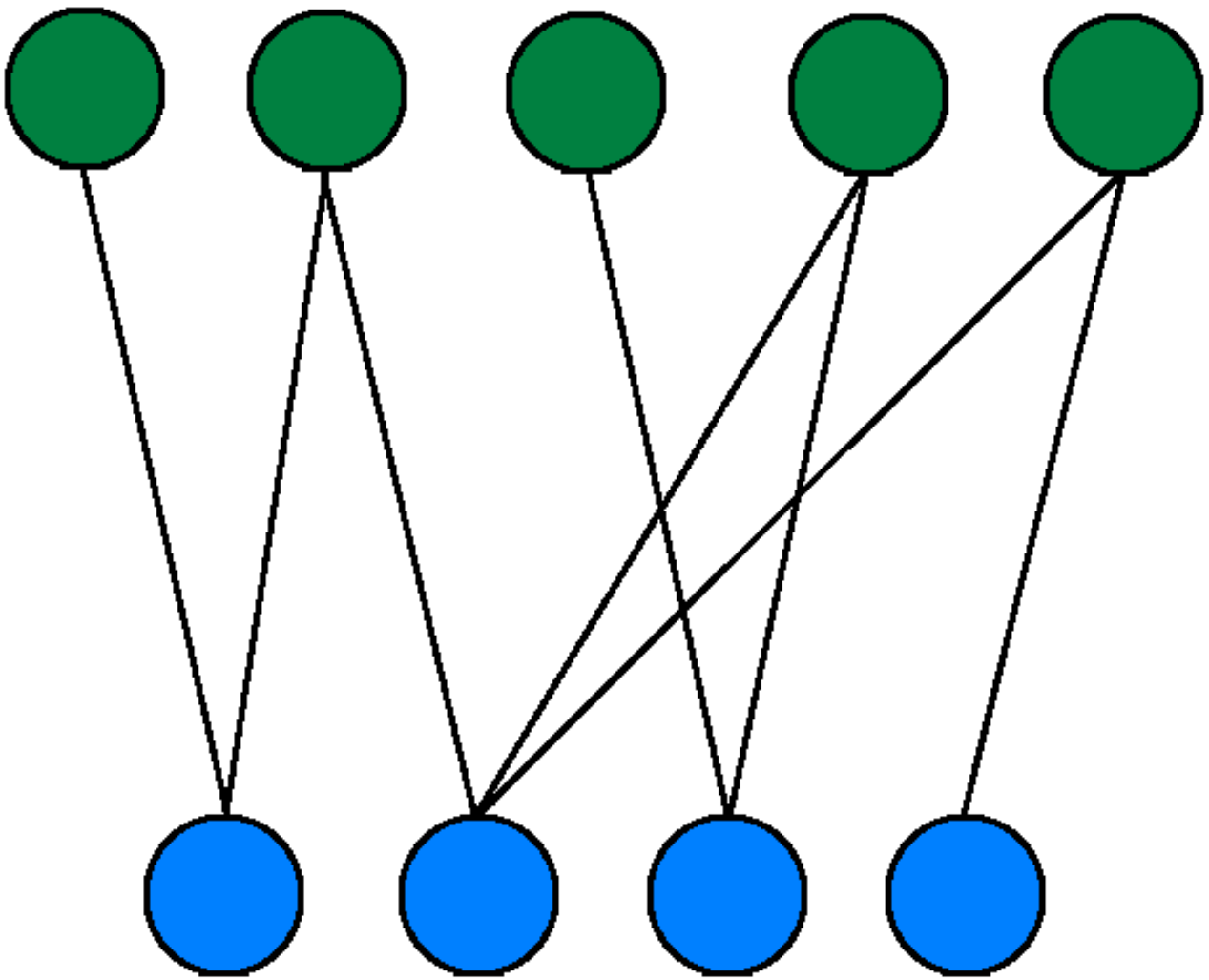

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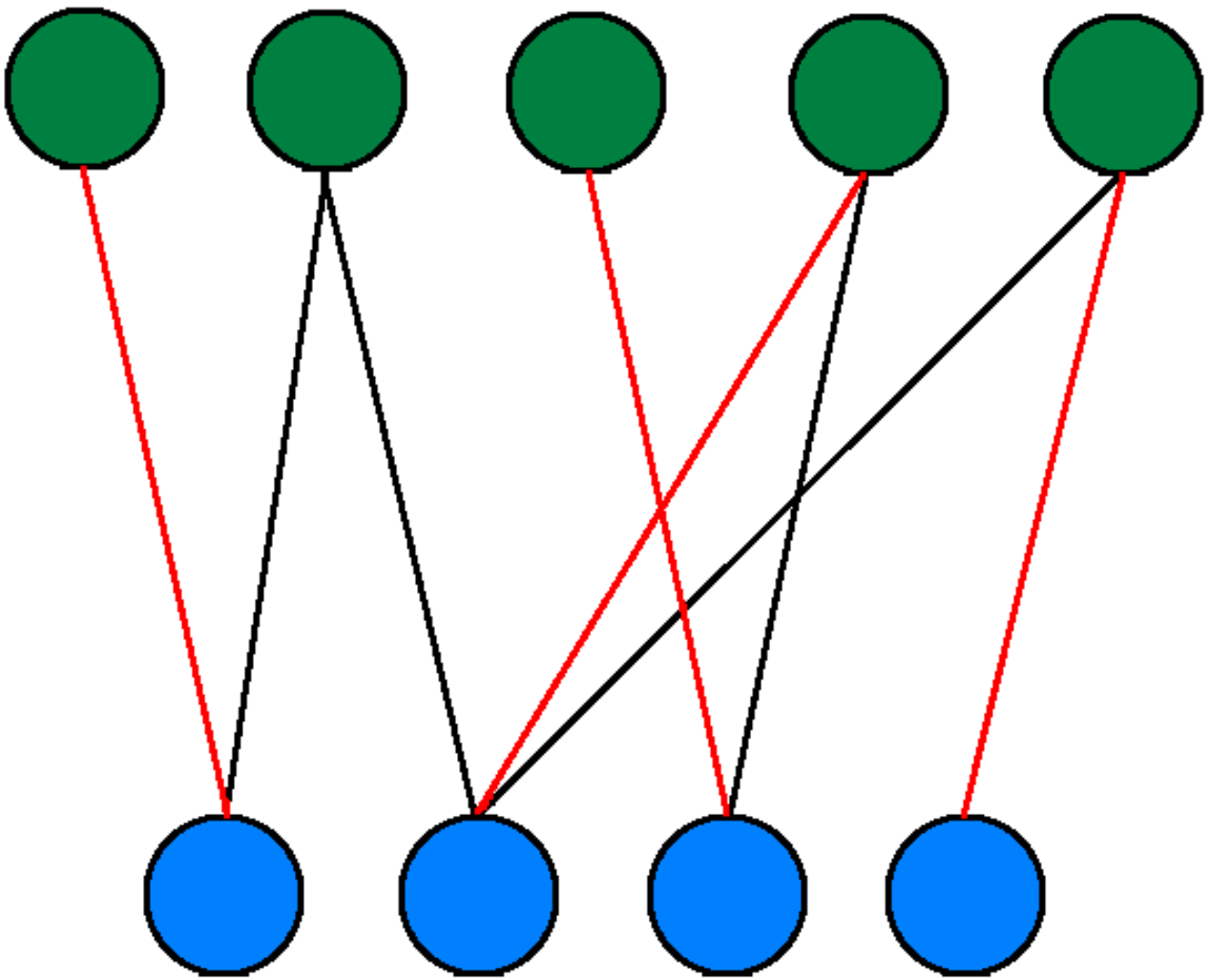

Some Notation

A marriage problem M consists of three sets B, G and R.

B is the set of boys,

G is the set of girls, and

R is the relation between the boys and girls.

$R \subset B \times G$ where $(b, g) \in R$ means "b knows g".

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$G_M(b)$ denotes the set of girls b knows relative to the relation in M.

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A *solution* to $M = (B, G, R)$ is an injection

$$f : B \to G$$

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M is a:

finite marriage problem if $|B|$ is finite.

infinite marriage problem if $|B|$ is not finite.

bounded marriage problem if there is a function $h : B \to G$ so that for each $b \in B$, $G(b) \subseteq \{0, 1, \ldots, h(b)\}$.

Examples of Marriage Theorems

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Theorem

If $M = (B, G, R)$ is a finite marriage problem such that $|G(B_0)| \geq |B_0|$ for every $B_0 \subset B$, then M has a solution.

Due to Philip Hall.

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(RCA_0) *If $M = (B, G, R)$ is a finite marriage problem with n boys and a unique solution f, then there is an enumeration of the boys $\langle b_i \rangle_{i \leq n}$ such that for every $1 \leq m \leq n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$.*

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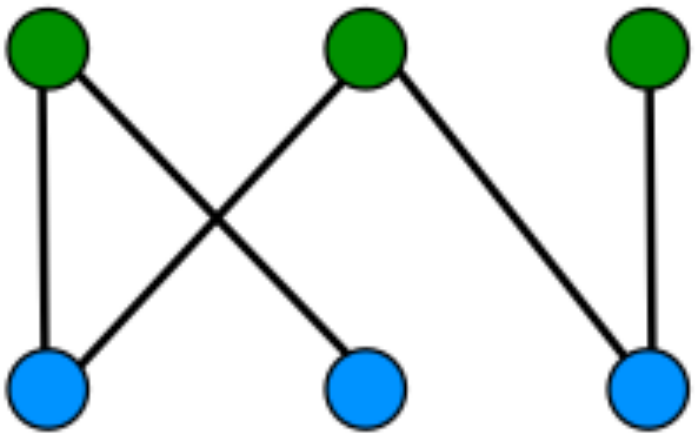

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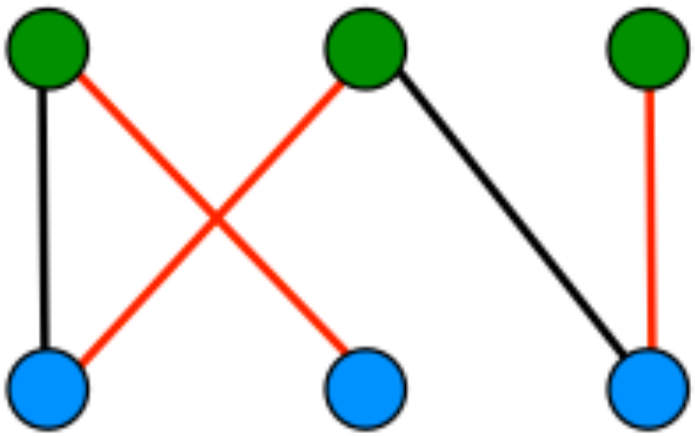

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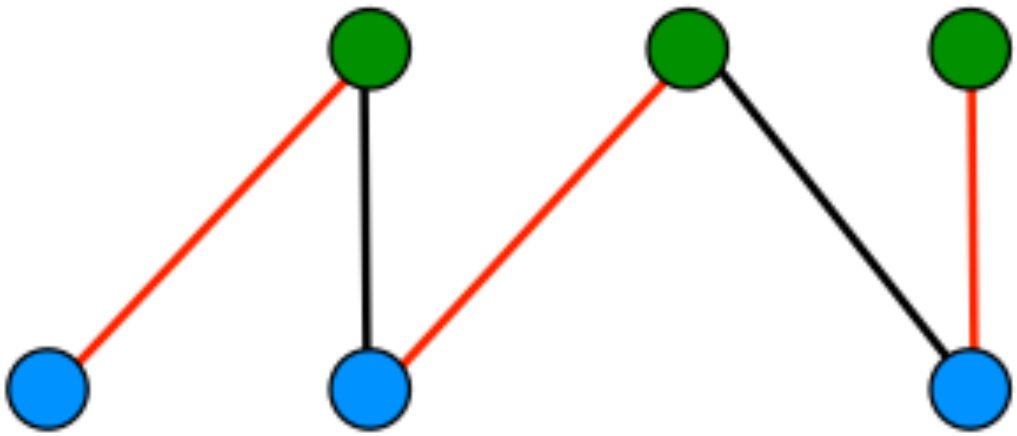

Sketch of the proof

Lemma

(RCA_0) *If* $M = (B, G, R)$ *is a finite marriage problem with a unique solution* f*, then some boy knows exactly one girl.*

Sketch of the proof

Proof: Suppose we have $M = (B, G, R)$ as stated above with some initial enumeration of B. Apply the lemma and let b_1 be the first boy such that $|G(b_1)| = 1$.

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Define $M_2 = (B - \{b_1\}, G - G(b_1), R_2)$. Because M has a unique solution, M_2 has a unique solution, namely the restriction of f to the sets of M_2. Apply the lemma once more and let b_2 be the first boy in $B - \{b_1\}$ such that $|G_{M_2}(b_2)| = 1$.

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Continuing this process inductively yields the jth boy in our desired enumeration from
$M_j = (B - \{b_1, b_2, \ldots, b_{j-1}\}, G - G(b_1, b_2, \ldots, b_{j-1}), R_j)$.

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After the nth iteration we have $(b_1, b_2, \ldots, b_n)$ where for every $1 \leq m \leq n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$. ■

Generalizing this result

The statement regarding finite marriage problems with unique solutions can be generalized to the *infinite* case. Paralleling the previous work we have:

Theorem

If $M = (B, G, R)$ is an infinite marriage problem with a unique solution f, then there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that for every $n \geq 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$.

II: Reverse Mathematics

Reverse Mathematics

Reverse mathematics is the subfield of mathematical logic dedicated to classifying the logical strength of mathematical theorems.

This is done by proving theorems equivalent to a hierarchy of axioms over a weak base axiom system.

$$\mathrm{RCA}_0 \qquad \mathrm{WKL}_0 \qquad \mathrm{ACA}_0 \qquad \mathrm{ATR}_0 \qquad \Pi^1_1 - \mathrm{CA}_0$$

RCA_0 proves the *intermediate value theorem* and the *uncountability of* $\mathbb{R}$.

RCA_0 does **not** prove the *existence of Riemann integrals.*

Equivalences

Theorem

The following are provable in RCA_0.

(i) $\mathsf{WKL}_0 \iff$ *For every continuous function* $f(x)$ *on a closed and bounded interval* $a \leq x \leq b$, *the Riemann integral* $\int_a^b f(x)dx$ *exists and is finite. (Simpson)*

(ii) $\mathsf{ACA}_0 \iff$ *For all one-to-one functions* $f : \mathbb{N} \to \mathbb{N}$ *there exists a set* $X \subseteq \mathbb{N}$ *such that* $\mathrm{Ran}(f) = X$. *(Simpson)*

(iii) $\mathsf{ATR}_0 \iff$ *Any two well orderings are comparable. (Friedman)*

(iv) $\Pi^1_1 - \mathsf{CA}_0 \iff$ *The* Cantor/Bendixson theorem for $\mathbb{N}^{\mathbb{N}}$: *Every closed set in* $\mathbb{N}^{\mathbb{N}}$ *is the union of a perfect closed set and a countable set. (Simpson)*

Marriage Theorems and Reverse Mathematics

Jeff Hirst proved the following equivalence results:

Theorem

(RCA_0) *If* $M = (B, G, R)$ *is a finite marriage problem such that* $|G(B_0)| \geq |B_0|$ *for every* $B_0 \subset B$*, then* M *has a solution.*

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(RCA_0) *The following are equivalent:*

1. ACA_0
2. *If $M = (B, G, R)$ is an infinite marriage problem such that $|G(B_0)| \geq |B_0|$ for every $B_0 \subset B$, then M has a solution.*

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Our new results echoed the previous work:

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Future Work

- Marriage problems with any fixed finite number of solutions.
- "Entangled societies"

References

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