Transversal Theory and Reverse Mathematics

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Transversals

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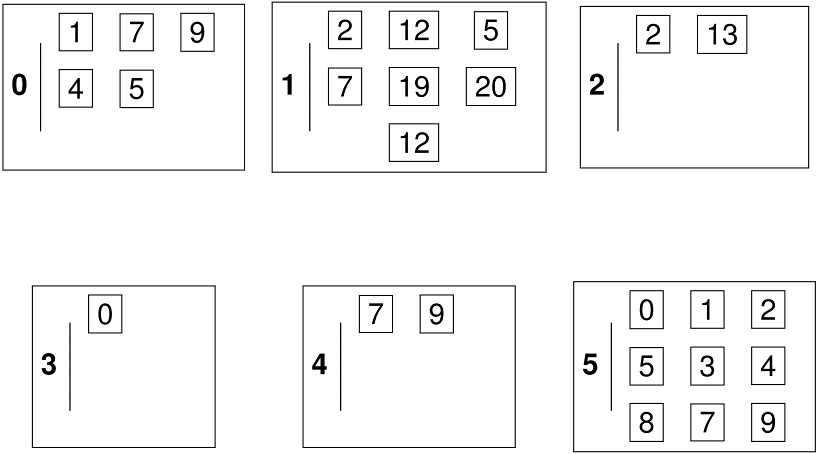

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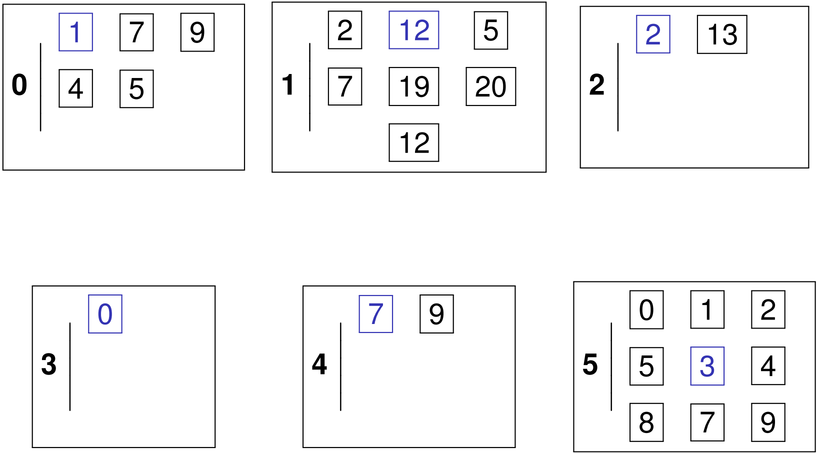

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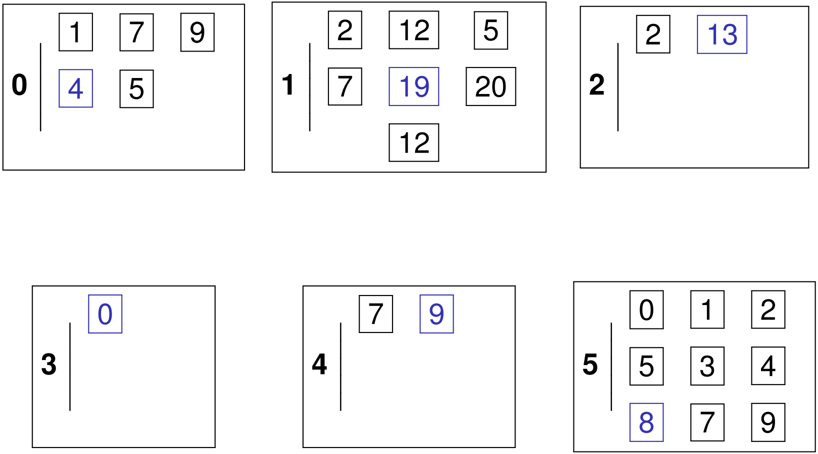

Transversals

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$$X_2 = \{2, 13\} \quad X_3 = \{0\}$$

$$X_4 = \{7, 9\} \quad X_5 = \{0, 1, 2, 5, 3, 4, 8, 7, 9\}$$

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$$T = \{4, 19, 13, 0, 9, 8\}$$

Previous Results for Transversals

Philip Hall and Marshal Hall Jr. (no relation) pioneered transversal theory.

Theorem

(Philip Hall's Theorem) *A collection of sets $X_0, X_1, \ldots, X_k$ has a transversal if and only if the union of any m sets has cardinality greater than or equal to m.*

Marshall Hall Jr. extended Philip Hall's work to the infinite case.

Theorem

(Marshall Hall's Theorem) *A collection of sets $X_0, X_1, \ldots$ has a transversal if and only if the union of any m sets has cardinality greater than or equal to m.*

Unique Solutions

What are the necessary and sufficient conditions for a collection of sets to have a *unique* transversal?

In the finite case, we found the following necessary and sufficient condition.

Theorem

(RCA_0) *A collection of sets $X_0, X_1, \ldots, X_k$ has a unique transversal if and only if there exists an enumeration of the sets $\langle X_i' \rangle_{i \leqslant k}$ such that for every $0 \leqslant j \leqslant k$, $|X_0' \cup X_1' \cup \cdots \cup X_j'| = j$.*

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Example:

$$\begin{array}{l} X_0 = \{0\} \\ X_1 = \{2,3\} \\ X_2 = \{2\} \end{array} \implies \begin{array}{l} X_0 = \{0\} \\ X_1 = \{2,3\} \\ X_2 = \{2\} \end{array} \qquad \begin{array}{l} X_0 = X_0' \\ X_1 = X_2' \\ X_2 = X_1' \end{array}$$

Reverse Mathematics

Reverse mathematics is the subfield of mathematical logic dedicated to classifying the logical strength of mathematical theorems.

This is done by proving theorems equivalent to a hierarchy of axioms over a weak base axiom system.

$$\mathrm{RCA}_0 \qquad \mathrm{WKL}_0 \qquad \mathrm{ACA}_0 \qquad \mathrm{ATR}_0 \qquad \Pi^1_1-\mathrm{CA}_0$$

RCA_0 proves the *intermediate value theorem* and the *uncountability of* $\mathbb{R}$.

RCA_0 does **not** prove the *existence of Riemann integrals.*

Equivalences

Theorem

The following are provable in RCA_0.

(i) $\mathrm{WKL}_0 \iff$ *For every continuous function* $f(x)$ *on a closed bounded interval* $a \leqslant x \leqslant b$, *the Riemann integral* $\int_a^b f(x)dx$ *exists and is finite. (Simpson)*

(ii) $\mathrm{ACA}_0 \iff$ *For all one-to-one functions* $f : \mathbb{N} \to \mathbb{N}$ *there exists a set* $X \subseteq \mathbb{N}$ *such that* $\mathrm{Ran}(f) = X$. *(Simpson)*

(iii) $\mathrm{ATR}_0 \iff$ *Any two well orderings are comparable. (Friedman)*

(iv) $\Pi^1_1 - \mathrm{CA}_0 \iff$ *The* Cantor/Bendixson theorem for $\mathbb{N}^{\mathbb{N}}$: *Every closed set in* $\mathbb{N}^{\mathbb{N}}$ *is the union of a perfect closed set and a countable set. (Simpson)*

Results

Jeff Hirst proved that Philip Hall's theorem is provable in RCA_0.

Theorem

(RCA_0) *A collection of sets $X_0, X_1, \ldots, X_k$ has a transversal if and only if the union of any m sets has cardinality greater than or equal to m.*

We found the enumeration theorem is provable in RCA_0 as well.

Theorem

(RCA_0) *A collection of sets $X_0, X_1, \ldots, X_k$ has a unique transversal if and only if there exists an enumeration of the sets $\langle X_i' \rangle_{i \leqslant k}$ such that for every $0 \leqslant j \leqslant k$, $|X_0' \cup X_1' \cup \cdots \cup X_j'| = j$.*

Results

Hirst also showed that Marshall Hall's theorem was provably equivalent to ACA_0 over RCA_0.

We found that the enumeration theorem for infinite collections of sets was also equivalent to ACA_0.

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0
2. *A collection of sets* $X_0, X_1, \ldots$ *has a unique transversal if and only if there exists an enumeration of the sets* $\langle X_i' \rangle_{i \geqslant 0}$ *such that for every* $0 \leqslant j$, $|X_0' \cup X_1' \cup \cdots \cup X_j'| = j$.

Sketch of the reversal

We assume statement (2) in order to prove statement (1). By Lemma III.1.3 of Simpson [3], it suffices to show (2) implies the existence of the range of an arbitrary injection.

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To that end, let $f : \mathbb{N} \to \mathbb{N}$ be an injection and construct the following sets:

- $X_i = \{(0, i)\}$ and $Y_i = \{(i, 0)\}$ for every $i \in \mathbb{N}$ and,
- if $f(m) = n$ then $(m, 0) \in X_n$, that is, $X_n = \{(m, 0), (0, n)\}$.

This collection obviously has a *unique transversal* consisting of $(0, i)$ from each X_i and $(i, 0)$ from each Y_i.

These coordinate pairs are encoded as natural numbers via the pairing map: $(i, j) = (i + j)^2 + i$.

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Well, this implies k is in the range of f if and only if some set Y_j appears before X_k in the enumeration and $f(j) = k$.

We need only check finitely many values of f to see if k is in the range, hence, recursive comprehension proves the existence of the range of f.

$\square$

An Open Question

To prove the enumeration theorem for infinite marriage problems we employed the following lemma.

Lemma

Suppose a collection of sets $C = X_0, X_1, \ldots$ has a unique transversal. Then for any set X_i there is a finite collection of sets S such that $X_i \in S \subset C$ and

$$|S| = \left| \bigcup_{X_j \in S} X_j \right|.$$

The exact strength of this statement is still unknown.

References

[1] Jeffry L. Hirst, *Marriage theorems and reverse mathematics*, Logic and computation (Pittsburgh, PA, 1987), Contemp. Math., vol. 106, Amer. Math. Soc., Providence, RI, 1990, pp. 181–196. DOI 10.1090/conm/106/1057822. MR1057822 (91k:03141)

[2] Jeffry L. Hirst and Noah A. Hughes, *Reverse mathematics and marriage problems with unique solutions*. Submitted.

[3] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. DOI 10.1017/CBO9780511581007 MR2517689.

Questions?

Thank You.