An Introduction to Reverse Mathematics

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Outline

- Preliminary Definitions
- Motivations
- Reverse mathematics
- Constructing the *big five* subsystems
- Original results regarding *marriage theorems*

Preliminaries

An **axiom system** is a set of mathematical statements we take as true. We then use the *axioms* to deduce mathematical theorems.

Example: ZFC is the standard foundation for mathematics.

Example: The *Peano axioms* are nine statements which define the natural numbers.

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Example: The *Peano axioms* are nine statements which define the natural numbers.

If we can prove a theorem φ in an axiom system $\mathfrak{T}$ then we write

$$\mathfrak{T} \vdash \varphi.$$

If φ requires an additional axiom A (along with those in $\mathfrak{T}$) to be proven we write

$$\mathfrak{T} + A \vdash \varphi \qquad \longleftrightarrow \qquad \mathfrak{T} \vdash A \Rightarrow \varphi.$$

Preliminaries

We build **formulas** from the three atomic formula

$$x = y, x < y, x \in X,$$

using logical connectives and quantifiers.

Logical Connectives:

$$\rightarrow, \leftrightarrow, \neg, \bigwedge, \bigvee$$

Quantifiers:

$$\exists x\,, \forall y\,, \exists X\,, \forall Y$$

<u>Example:</u>

$$x \in X \leftrightarrow \exists y\,(x = 2 \cdot y).$$

A Question

How do theorems relate in mathematics?

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Suppose we have two mathematical theorems φ_1 and φ_2 that we would like to compare.

$\rightarrow$ What does it mean to say φ_1 is “stronger” than φ_2?

$\rightarrow$ Or to say φ_1 and φ_2 are “equivalent”?

$\rightarrow$ Can we determine if these theorems are even *comparable* or are they independent of each other?

$\rightarrow$ What if φ_1 and φ_2 are from different areas of mathematics?

A Possible Strategy

Suppose we have a substantially weak axiom system $\mathfrak{B}$ (the *base theory*) that proves φ_1 but not does not prove φ_2:

$$\mathfrak{B} \vdash \varphi_1 \qquad\qquad \mathfrak{B} \nvdash \varphi_2.$$

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This is a rough measure of logical strength. A_1 may be wildly powerful and give us little insight into the difference in φ_1 and φ_2

“Reversing” mathematics for a better measure

Because $\mathfrak{B} + A_1 \vdash \varphi_2$ we already know

$$\mathfrak{B} \vdash A_1 \Rightarrow \varphi_2.$$

Suppose we can show $\mathfrak{B} + \varphi_2 \vdash A_1$, that is,

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We can now conclude that A_1 and φ_2 are *provably equivalent* over the base theory $\mathfrak{B}$, *i.e.*

$$\mathfrak{B} \vdash A_1 \iff \varphi_2.$$

Extending this classification

Let's consider a third theorem φ_3.

Suppose after some analysis we find another axiom A_2 differing from A_1 such that

$$\mathfrak{B} \vdash A_2 \iff \varphi_3.$$

What can we conclude about the relationships between our three theorems φ_1, φ_2 and φ_3?

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To determine the relationship between φ_2 and φ_3 we need to know how A_1 and A_2 compare.

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Possible complications:

- It may be extremely difficult to determine the relationship between two axioms.
- The theorems of mathematics are extremely diverse. As we consider more theorems we may need more and more axioms to determine their logical strength.
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In short, this could become a real mess.

It is! (Surprisingly)

It turns out that with the specific base theory RCA_0 we need only **four** additional axioms (A_1, A_2, A_3, A_4) to classify an *enormous* amount of mathematical theorems.

We call RCA_0 and the four axiom systems which are obtained from appending A_1, A_2, A_3 or A_4 to the base theory the *big five:*

$$\mathrm{RCA}_0 \qquad \mathrm{WKL}_0 \qquad \mathrm{ACA}_0 \qquad \mathrm{ATR}_0 \qquad \Pi^1_1-\mathrm{CA}_0.$$

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Reverse mathematics is the program dedicated to classifying the logical strength of mathematical theorems via these five axiom systems.

Reverse mathematics

$$\mathrm{RCA}_0 \qquad \mathrm{WKL}_0 \qquad \mathrm{ACA}_0 \qquad \mathrm{ATR}_0 \qquad \Pi^1_1 - \mathrm{CA}_0$$

Each is a *weak* subsystem of *second order arithmetic*.

The strength of each system is measured by the amount of *set comprehension* available.

Example: Take our three theorems φ_1, φ_2 and φ_3. If we show

$$\mathrm{RCA}_0 \vdash \varphi_1$$

$$\mathrm{RCA}_0 \vdash \mathrm{WKL}_0 \iff \varphi_2$$

$$\mathrm{RCA}_0 \vdash \mathrm{ACA}_0 \iff \varphi_3,$$

we know the theorems compare in terms of logical strength.

Second Order Arithmetic

Denoted $\mathbf{Z}_2$.

Language:
Number variables: x, y, z Set variables: X, Y, Z

basic arithmetic axioms

$$n+1 \neq 0$$
$$m+1 = n+1 \rightarrow m = n$$
$$m+0 = m$$
$$m+(n+1) = (m+n)+1$$
$$m \cdot 0 = 0$$
$$m \cdot (n+1) = (m \cdot n) + m$$
$$\neg\, m < 0$$
$$m < n+1 \leftrightarrow (m < n \vee m = n)$$

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The second order induction scheme
$(\psi(0) \wedge \forall n\,(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n\,\psi(n)$
where $\psi(n)$ is any *formula* in $\mathbf{Z}_2$.

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where $\psi(n)$ is any *formula* in $\mathbf{Z}_2$.

Set comprehension

$$\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of $\mathbf{Z}_2$ in which X does not occur freely.

Recursive Comprehension and RCA_0

RCA_0 is the subsystem of $\mathbf{Z}_2$ whose axioms are:

basic arithmetic axioms

Restricted induction

$$(\psi(0) \wedge \forall n\,(\psi(n) \rightarrow \psi(n+1))) \rightarrow \forall n\,\psi(n)$$

where $\psi(n)$ has (at most) one number quantifier.

Recursive set comprehension

Recursive or *computable* sets exist.

Coding

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The *pairing map*:

$$(i, j) = (i + j)^2 + i.$$

This encodes pairs as a single natural number:

$$(2, 3) = (2 + 3)^2 + 2 = 27 \qquad (0, 17) = (0 + 17)^2 + 0 = 17^2$$

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Encoding a triple:
$(2, 3, 4) = ((2, 3), 4) = (27, 4) = (27 + 4)^2 + 27$

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Within RCA_0 we may construct the number system of the integers $\mathbb{Z}$.

Using the pairing map, we identify the (code for the) pair (m, n) with the integer $m - n$.

To define arithmetic on $\mathbb{Z}$ we make several definitions for "$\mathbb{Z}$ arithmetic" on these pairs.

$$\begin{aligned}
(m,n) +_{\mathbb{Z}} (p,q) &= (m+p, n+q) \\
(m,n) -_{\mathbb{Z}} (p,q) &= (m+q, n+p) \\
(m,n) \cdot_{\mathbb{Z}} (p,q) &= (m \cdot p + n \cdot q, m \cdot q + n \cdot p) \\
(m,n) <_{\mathbb{Z}} (p,q) &\leftrightarrow m+q < n+p \\
(m,n) =_{\mathbb{Z}} (p,q) &\leftrightarrow m+q = n+p
\end{aligned}$$

Coding

We can encode much more within RCA_0, including:

- The rational numbers.
- Real numbers.
- Countable abelian groups and vector spaces.
- Continuous real-valued functions.
- Complete separable metric spaces.

How strong is RCA_0?

Theorem

The following are provable in RCA_0*.*

(i) *The system* $\mathbb{Q}, +, -, \cdot, 0, 1, <$ *is an ordered field. (Simpson)*

(ii) *The uncountability of* $\mathbb{R}$*. (Simpson)*

(iii) *The intermediate value theorem on continuous real-valued functions. If* $f(x)$ *is a continuous real-valued function on the unit interval* $0 \leqslant x \leqslant 1$ *and* $f(0) < 0 < f(1)$*, then there exists* c *such that* $0 < c < 1$ *and* $f(c) = 0$*. (Simpson)*

(iv) *Basics of real linear algebra, including Gaussian Elimination. (Simpson)*

How strong is RCA_0?

Theorem

The following are **not** *provable in* RCA_0*.*

(i) *The maximum principle: Every continuous real-valued function on* $[0,1]$ *attains a supremum. (Simpson)*

(ii) *For every continuous function* $f(x)$ *on a closed bounded interval* $a \leqslant x \leqslant b$*, the Riemann integral* $\int_a^b f(x)dx$ *exists and is finite. (Simpson)*

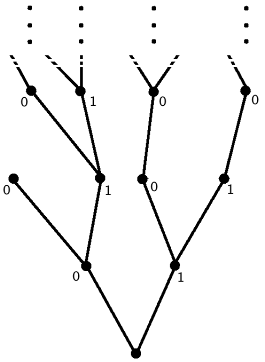
So we see RCA_0 does not prove everything. This is desirable.

Weak König's Lemma and WKL_0

The next subsystem of $\mathbf{Z}_2$ is obtained by appending *weak König's lemma* to RCA_0.

Weak König's lemma states that:

If T is an infinite binary tree, then T contains an infinite path.



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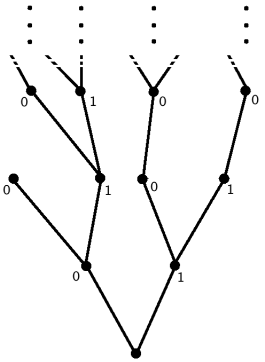

So weak König's lemma basically says:

"Big, skinny trees are tall."

$\mathsf{RCA}_0 \nvdash$ weak König's lemma

How strong is WKL_0?

Theorem

One can prove the following statements equivalent to WKL_0 *over* RCA_0.

(i) *The maximum principle: Every continuous real-valued function on* $[0, 1]$ *attains a supremum. (Simpson)*

(ii) *Every continuous real-valued function on* $[0, 1]$ *is bounded. (Simpson)*

(iii) *For every continuous function* $f(x)$ *on a closed bounded interval* $a \leqslant x \leqslant b$*, the Riemann integral* $\int_a^b f(x)dx$ *exists and is finite. (Simpson)*

(iv) *Every countable field has a unique algebraic closure. (Friedman, Simpson, and Smith)*

(v) Peano's existence theorem *for solutions to ODEs. (Simpson)*

Arithmetical Comprehension and ACA_0

ACA_0 is RCA_0 plus comprehension for arithmetically definable sets.

The arithmetical comprehension scheme:

> For any formula $\theta(n)$ with only number quantifiers, the set $\{n \in \mathbb{N} \mid \theta(n)\}$.

Note: $\mathsf{WKL}_0 \nvdash \mathsf{ACA}_0$ but $\mathsf{ACA}_0 \vdash \mathsf{WKL}_0$.

How strong is ACA_0?

Theorem

One can prove the following statements equivalent to ACA_0 *over* RCA_0.

(i) *Cauchy sequences converge. (Simpson)*

(ii) *The Bolzano/Weierstraß theorem: Every bounded sequence of real numbers contains a convergent subsequence. (Friedman)*

(iii) *The Ascoli lemma. (Simpson)*

(iv) *Ramsey's theorem for triples. (Simpson)*

Arithmetical Transfinite Recursion and ATR_0

ATR_0 consists of RCA_0 plus axioms which allow for iteration of arithmetical comprehension along any well ordering. This allows transfinite constructions.

This system is *vastly* stronger than ACA_0.

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This system is *vastly* stronger than ACA_0.

Theorem

One can prove the following statements equivalent to ATR_0 *over* RCA_0*.*

(i) *Any two well orderings are comparable. (Friedman)*

(ii) *Every countable reduced Abelian p-group has an Ulm resolution. (Friedman, Simpson, and Smith)*

(iii) Sherman's Inequality: *If* α, β *and* γ *are countable well orderings, then* $(\alpha + \beta)\gamma \leqslant \alpha\gamma + \beta\gamma$*. (Hirst)*

Π^1_1 Comprehension and $\Pi^1_1 - \mathsf{CA}_0$

The system $\Pi^1_1 - \mathsf{CA}_0$ consists of RCA_0 plus comprehension for Π^1_1 definable sets. That is, we can assert the existence of the set

$$\{n \in \mathbb{N} \mid \theta(n)\}$$

where θ is a Π^1_1 formula, meaning θ has one universal set quantifier $(\forall X)$ and no other set quantifiers.

Theorem

The following are provably equivalent to $\Pi^1_1 - \mathsf{CA}_0$ *over* RCA_0.

(i) *The* Cantor/Bendixson theorem for $\mathbb{N}^{\mathbb{N}}$: *Every closed set in* $\mathbb{N}^{\mathbb{N}}$ *is the union of a perfect closed set and a countable set. (Simpson)*

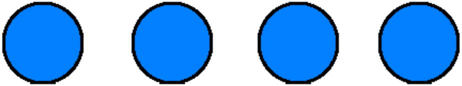
(ii) *Every countable Abelian group is the direct sum of a divisible group and a reduced group. (Friedman, Simpson, and Smith)*

Consequences of Reverse Math

- We can formalize many of the theorems in mathematics as one of only *five* statements.
- This makes the exceptions that much more interesting.
- Reverse math over stronger systems, *e.g.,* ZFC as the base theory.

Marriage Problems

Marriage Problems



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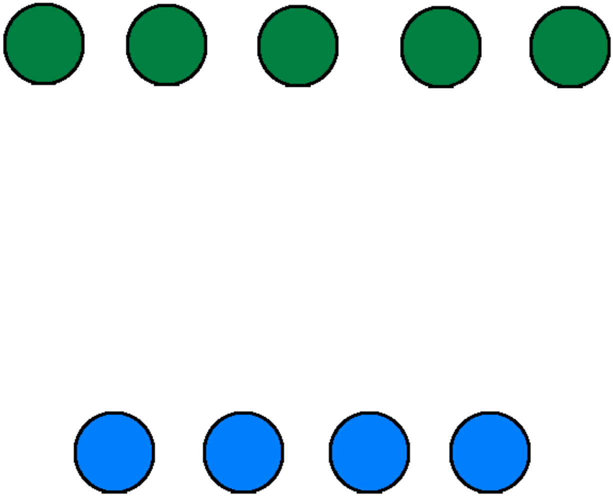

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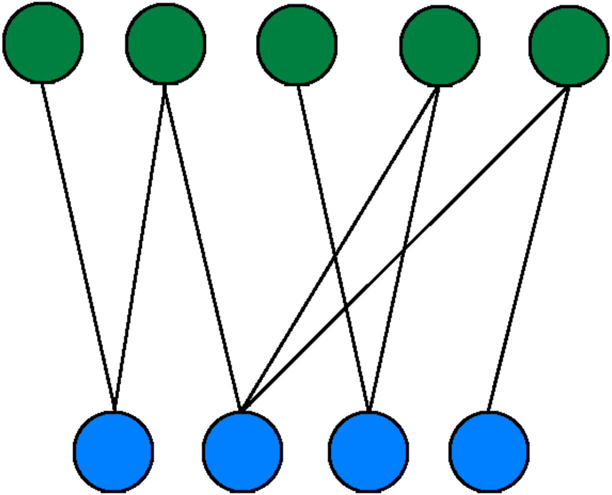

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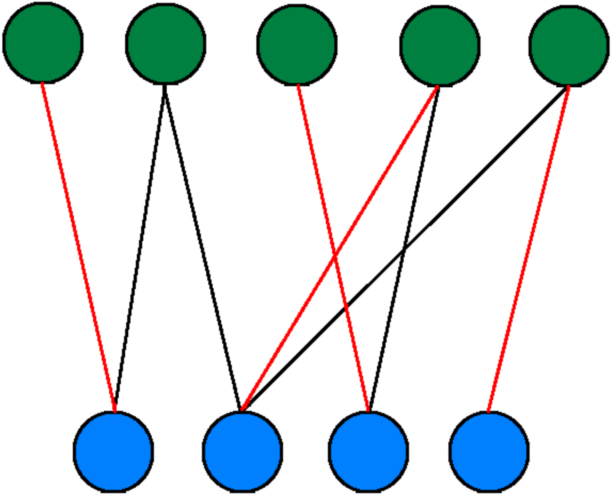

Some Notation

A marriage problem M consists of three sets B, G and R.

B is the set of boys,

G is the set of girls, and

R is the relation between the boys and girls.

$R \subset B \times G$ where $(b, g) \in R$ means "b knows g".

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$R \subset B \times G$ where $(b, g) \in R$ means "b knows g".

$G(b)$ is convenient shorthand for the set of girls b knows, *i.e.*

$$G(b) = \{g \in G \mid (b, g) \in R\}.$$

$G(b)$ is *not* a function.

Some More Notation

A *solution* to $M = (B, G, R)$ is an injection

$$f : B \to G$$

such that $(b, f(b)) \in R$ for every $b \in B$.

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M is a:

finite marriage problem if $|B|$ is finite.

infinite marriage problem if $|B|$ is not finite.

bounded marriage problem if there is a function $h : B \to G$ so that for each $b \in B$, $G(b) \subseteq \{0, 1, \ldots, h(b)\}$.

Previous Work

Jeff Hirst showed the following theorem of Philip Hall is provable within RCA_0.

Theorem

(RCA_0) *If $M = (B, G, R)$ is a finite marriage problem such that $|G(B_0)| \geqslant |B_0|$ for every $B_0 \subset B$, then M has a solution.*

Marshall Hall Jr. extended Philip Hall's work to the infinite case.

Theorem

If $M = (B, G, R)$ is an infinite marriage problem where each boy knows only finitely many girls and $|G(B_0)| \geqslant |B_0|$ for every $B_0 \subset B$, then M has a solution.

Previous Work

Hirst proved the following equivalence results.

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0
2. *If $M = (B, G, R)$ is an infinite marriage problem where each boy knows only finitely many girls and $|G(B_0)| \geqslant |B_0|$ for every $B_0 \subset B$, then M has a solution.*

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0
2. *If $M = (B, G, R)$ is a bounded marriage problem such that $|G(B_0)| \geqslant |B_0|$ for every $B_0 \subset B$, then M has a solution.*

Unique Solutions

What are the necessary and sufficient conditions for a marriage problem to have a *unique* solution?

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In the finite case, we found the following necessary and sufficient condition.

Theorem

(RCA_0) *If $M = (B, G, R)$ is a finite marriage problem with n boys a unique solution f, then there is an enumeration of the boys $\langle b_i \rangle_{i \leqslant n}$ such that for every $1 \leqslant m \leqslant n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$.*

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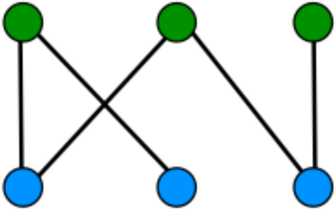

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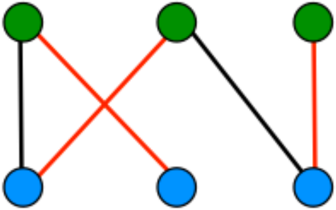

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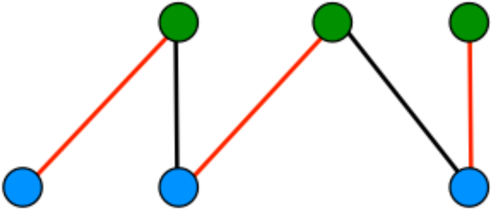

Sketch of the proof

Lemma

(RCA_0) *If $M = (B, G, R)$ is a finite marriage problem with a unique solution f, then some boy knows exactly one girl.*

Sketch of the proof

Suppose we have $M = (B, G, R)$ as stated above with some initial enumeration of B. Apply the lemma and let b_1 be the first boy such that $|G(b_1)| = 1$.

Define $M_2 = (B - \{b_1\}, G - G(b_1), R_2)$. Because M has a unique solution, M_2 has a unique solution, namely the restriction of f to the sets of M_2. Apply the lemma once more and let b_2 be the first boy in $B - \{b_1\}$ such that $|G_{M_2}(b_2)| = 1$.

Continuing this process inductively yields the j^{th} boy in our desired enumeration from
$M_j = (B - \{b_1, b_2, \ldots, b_{j-1}\}, G - G(b_1, b_2, \ldots, b_{j-1}), R_j)$.

After the n^{th} iteration we have $(b_1, b_2, \ldots, b_n)$ where for every $1 \leqslant m \leqslant n$, $|G(\{b_1, b_2, \ldots, b_m\})| = m$. $\square$

Infinite Marriage Problems

The statement regarding finite marriage problems with unique solutions can be generalized to the *infinite* case. Paralleling the previous work we see:

Theorem

(RCA_0) *The following are equivalent:*

1. ACA_0
2. *If $M = (B, G, R)$ is an infinite marriage problem where each boy knows only finitely many girls and has a unique solution f, then there is an enumeration of the boys $\langle b_i \rangle_{i \geqslant 1}$ such that for every $n \geqslant 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$.*

Sketch of the reversal

We assume statement (2) in order to prove statement (1). By Lemma III.1.3 of Simpson [3], it suffices to show (2) implies the existence of the range of an arbitrary injection.

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To that end, let $f : \mathbb{N} \to \mathbb{N}$ be an injection and construct $M = (B, G, R)$ as follows:

- $B = \{c_n \mid n \in \mathbb{N}\} \cup \{d_n \mid n \in \mathbb{N}\}$,
- $G = \{g_n \mid n \in \mathbb{N}\} \cup \{r_n \mid n \in \mathbb{N}\}$,
- for every i, $(c_i, g_i) \in R$ and $(d_i, r_i) \in R$, and
- if $f(m) = n$ then $(c_n, r_m) \in R$.

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Let $h : B \to G$ such that $h(d_i) = r_i$ and $h(c_i) = g_i$ for each $i \in \mathbb{N}$.

h is injective and a unique solution to M.

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Apply the enumeration theorem to obtain $\langle b_i \rangle_{i \geqslant 1}$ where for every $n \geqslant 1$ $|G(b_1, \ldots, b_n)| = n$.

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Well, this implies that k is in the range of f if and only if some boy d_j appears before c_k in the enumeration and $f(j) = k$.

We need only check finitely many values of f to see if k is in the range, hence, recursive comprehension proves the existence of the range of f.

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Bounded Marriage Problems

In the *bounded* case, the result, as expected, paralleled the previous work.

Theorem

(RCA_0) *The following are equivalent:*

1. WKL_0
2. *If* $M = (B, G, R)$ *is a bounded marriage problem with a unique solution* f*, then there is an enumeration of the boys* $\langle b_i \rangle_{i \geqslant 1}$ *such that for every* $n \geqslant 1$, $|G(\{b_1, b_2, \ldots, b_n\})| = n$.

An Open Question

To prove the enumeration theorem for infinite marriage problems we employed the following lemma.

Lemma

Suppose $M = (B, G, R)$ is a marriage problem with a unique solution, then for any $b \in B$ there is a finite set F such that $b \in F \subset B$ and $|G(F)| = |F|$.

The exact strength of this statement is still unknown.

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Questions?

Thank You.