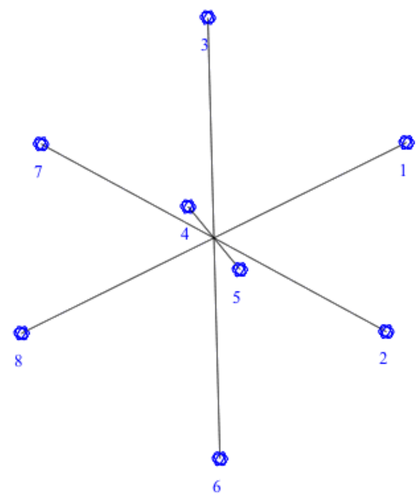
Permutations and Weyl Groups: Seeing Irreducibility in Cycle Structures



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Outline:

* A brief introduction to finite simple Lie Algebras

* Root Systems, Symmetries and Weyl groups

* Minuscule Representation through permutation sets

* Results of our research

Definition:

Let V be a vector space (over $\mathbb{C}$) equipped with a bilinear multiplication (called a "bracket")

$$[\,\cdot\,,\,\cdot\,] : V \times V \to V$$

such that,

$$\text{(Jacobi Identity)} \quad [\,u,\,[v,w]\,] = [\,[u,v],\,w\,] + [\,v,\,[u,w]\,] \quad \forall\, u,\, v,\, w \in V$$

and

$$\text{(Alternating)} \qquad [\,v,\,v\,] = 0 \qquad \forall\, v \in V$$

or equivalently

$$\text{(Skew symmetry)} \qquad [\,v,\,w\,] = -[\,w,\,v\,] \qquad \forall\, v \in V$$

Then V is a *Lie algebra*.

Example:

The general linear Lie algebra: $\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ where $[A,B] = AB - BA$ (the commutator bracket)

Definition:

Let V be a Lie algebra. Then V is called a *simple* Lie algebra if it is not "Abelian" [meaning there exists $v, w \in V$ such that $[v, w] \neq 0$] and V has no non-trivial proper ideals.

$\Longrightarrow$ Simple Lie algebras are essentially the atomic building blocks of all Lie algebras.

Around 1900, Killing and Cartan found and classified all *finite dimensional simple* Lie algebras (over $\mathbb{C}$).

They labeled each as follows:

The classical algebras:

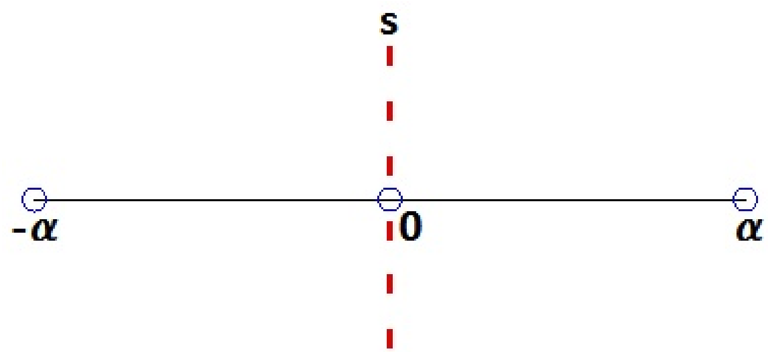
$$A_n = \mathfrak{sl}_{n+1} \quad (n \geq 1) \qquad C_n = \mathfrak{sp}_{2n} \quad (n \geq 3)$$

$$B_n = \mathfrak{so}_{2n+1} \quad (n \geq 2) \qquad D_n = \mathfrak{so}_{2n} \quad (n \geq 4)$$

The exceptional algebras:

$$E_6 \quad E_7 \quad E_8 \quad F_4 \quad \text{and} \quad G_2$$

The smallest simple Lie algebra: $A_1 = \mathfrak{sl}_2$



$A_1 = \mathfrak{sl}_2 = \{X \in \mathbb{C}^{2\times 2} \,|\, \mathrm{tr}(X) = 0\}$

Let $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad F = E^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad$ and $\quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\{E, F, H\}$ is a basis for $\mathfrak{sl}_2$

Note: $\quad [E, F] = EF - FE = H \qquad [H, E] = 2E \qquad [H, F] = -2F$

Theorem: Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra (over $\mathbb{C}$). There exists a subalgebra, $\mathfrak{h}$ (called a Cartan subalgebra), such that

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \,|\, [h, v] = \alpha(h)v \;\forall\, h \in \mathfrak{h}\}$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, then α is called a root of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is its root space.

Think "root = eigenvalue" and "root space = eigenspace"

Rank of $\mathfrak{g} = \dim(\mathfrak{h})$

A more involved example: $A_2 = \mathfrak{sl}_3$

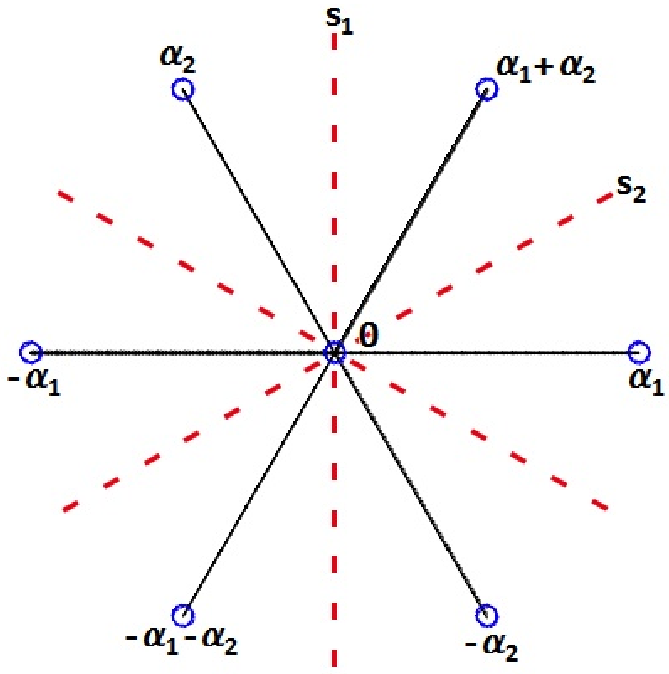
$$A_2 = \mathfrak{sl}_3 = \{X \in \mathbb{C}^{3\times 3} \,|\, \mathrm{tr}(X) = 0\}$$

Let $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and

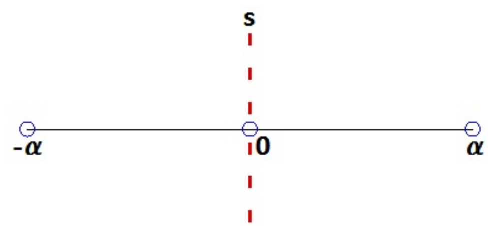
$$E_3 = [\,E_1,\, E_2\,] = E_1E_2 - E_2E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $F_1 = E_1^T \quad F_2 = E_2^T \quad F_3 = E_3^T$

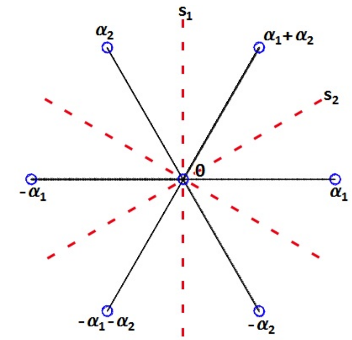
Let $H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$



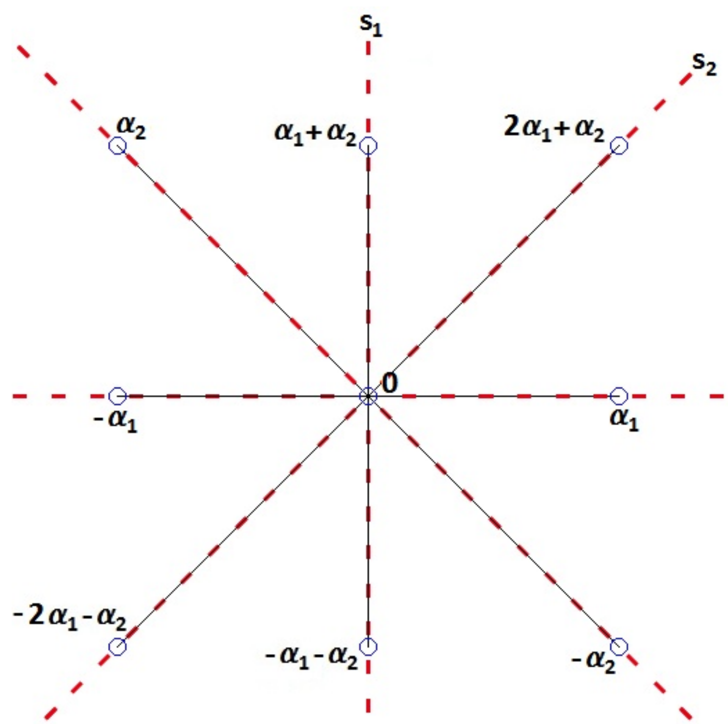
$\{E_1, E_2, E_3, H_1, H_2, F_1, F_2, F_3\}$ is a basis for $\mathfrak{sl}_3$



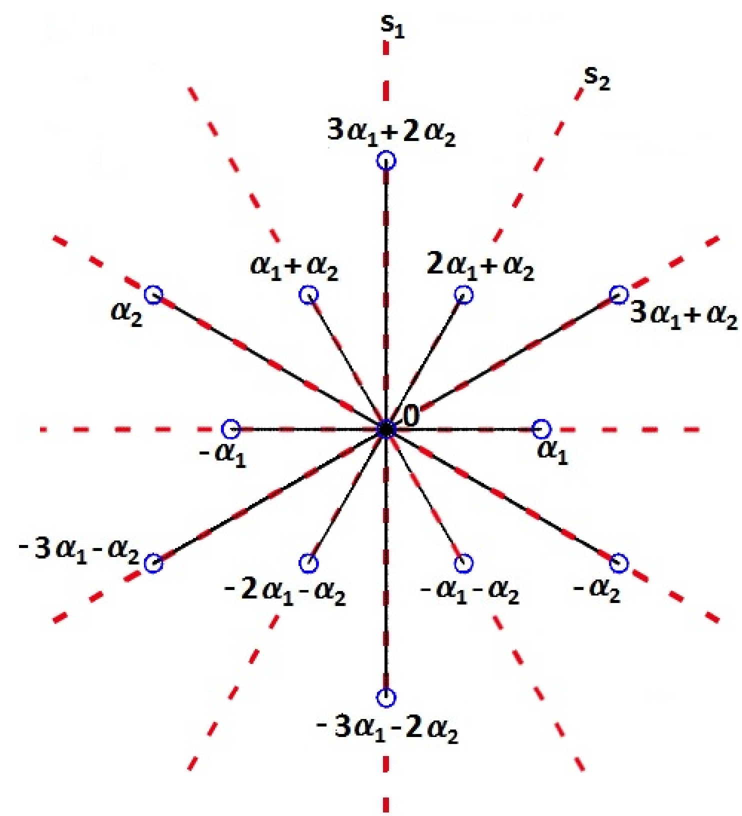
$A_1 = \mathfrak{sl}_2$'s root system



$A_2 = \mathfrak{sl}_3$'s root system



$B_2 = \mathfrak{so}_5$'s root system



G_2's root system

Weyl Group

Definition:

Let $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a set of simple roots for a finite dimensional simple Lie algebra, $\mathfrak{g}$. Define the simple reflection s_i to be the reflection across the hyperplane determined by α_i. The group of isometries generated by these simple reflections:

$$W = \langle s_1, s_2, ..., s_n \rangle$$

is called the Weyl group of $\mathfrak{g}$.

Example:

The Weyl group of $A_n = \mathfrak{sl}_{n+1}$ is S_{n+1} (the symmetric group). In this case $|W| = (n+1)!$.

The Weyl group of $B_n = \mathfrak{so}_{2n+1}$ is a semi-direct product of S_n and $(\mathbb{Z}_2)^n$. So $|W| = 2^n n!$.

Definition:

Let $\mathfrak{g}$ be a Lie algebra and V a vector space (over $\mathbb{C}$).
V is $\mathfrak{g}$-module if it is equipped with a bilinear action

$$\_ \cdot \_ : \mathfrak{g} \times V \to V \qquad \text{where} \qquad (g, v) \mapsto g \cdot v$$

and for all $x, y \in \mathfrak{g}$ and $v \in V$ we have

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

Definition:

A $\mathfrak{g}$-module V is irreducible if $V \neq \{0\}$ and V has no non-trivial proper submodules.

Examples:

Using regular matrix-vector multiplication, $\mathbb{C}^2$ becomes an irreducible $\mathfrak{sl}_2$-module.

Any simple Lie algebra $\mathfrak{g}$ acting on itself via $(g, x) \mapsto [g, x]$ is an irreducible $\mathfrak{g}$-module (this is called the adjoint module).

Theorem: Let $\mathfrak{g}$ be a finite dimensional simple Lie Algebra over $\mathbb{C}$
Let V be a finite dimensional irreducible $\mathfrak{g}$-module.
Then,

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

where

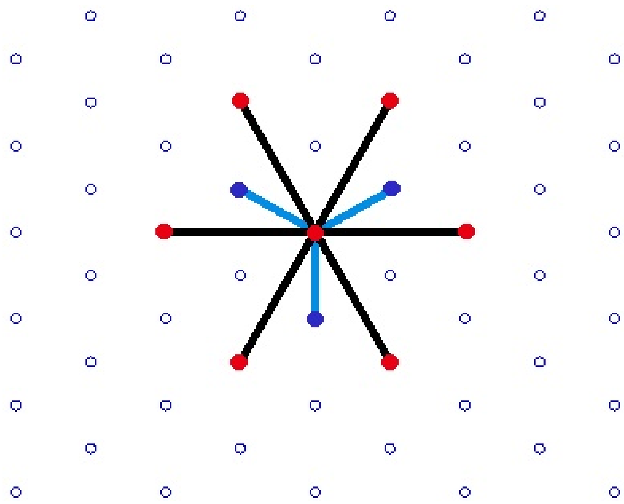
$$V_\lambda = \{v \in V | h \cdot v = \lambda(h)v \quad \forall h \in \mathfrak{h}\}$$

If $V_\lambda \neq \{0\}$, then λ is called a weight of V and V_λ is its weight space.

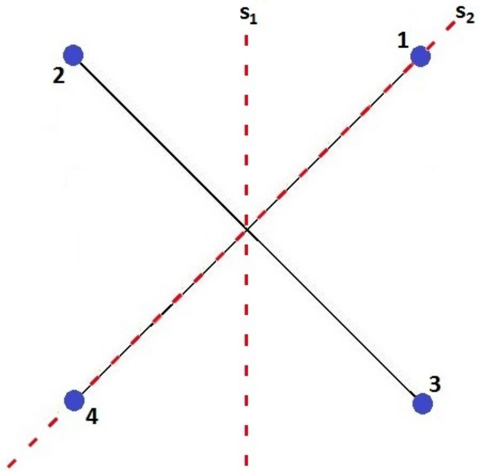
Again, think "weight = eigenvalue" and "weight space = eigenspace"

Definition:
A minuscule representation is an irreducible $\mathfrak{g}$-module whose weights all lie in a single Weyl group orbit (the Weyl group acts transitively on the set of weights).

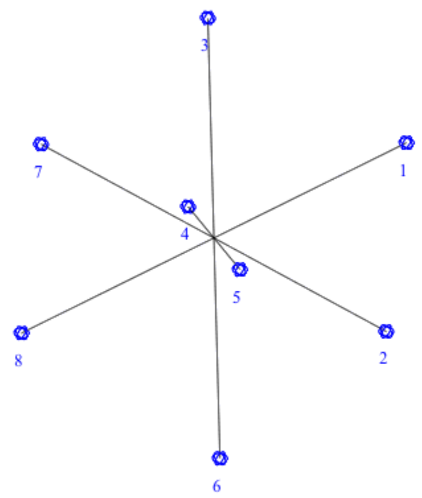
Examples of Minuscule Representations:



$A_2 = \mathfrak{sl}_3$'s weight lattice



$B_2 = \mathfrak{so}_5$'s min. rep.



$B_3 = \mathfrak{so}_7$'s min. rep.

The Project

Given a minuscule representation, let the Weyl group permute the weights of this module. Since we are viewing the elements of the Weyl group as permutations, we can speak of their cycle structures.

Question:

"Can we see our module's irreducibility from the Weyl group's cycle structures alone?"

Note: If there were more than one orbit of weights, the answer would automatically be "No". That is why we only consider minuscule representations.

Cook-Singer-Mitschi showed that one can see the irreducibility of minscule modules from their Weyl group cycle structures for all of the algebras except possibly type B_n. This summer along with Dr. Cook, I studied this remaining case. Our results were as follows:

B_2: $W = \Big\langle (12)(34), (23) \Big\rangle$. W has cycle structures:

$$1+1+1+1 \quad = \quad 1+1+2 \quad = \quad 2+2 \quad = \quad 4$$

The "4" guarantees irreducibility.

B_3: $W = \Big\langle (12)(34)(56)(78), (23)(67), (35)(46) \Big\rangle$ has cycle structures:

$$\begin{aligned} & 1+1+\cdots+1 = 1+1+1+1+2+2 = 1+1+3+3 \\ = \; & 2+2+2+2 = 2+6 = 4+4 \end{aligned}$$

Here $2+6$ only allows 0, 2, 6, or 8 dimensional submodules and $4+4$ only allows 0, 4, or 8 dimensional submodules so together they guarantee irreducibility.

B_4: $W = \Big\langle (12)(34)\cdots(15,16), (23)(67)(10,11)(14,15),$
$(35)(46)(11,13)(12,14), (59)(6,10)(7,11)(8,12)\Big\rangle.$

W has cycle structures:

$$\begin{array}{rcccccc} & 1+1+\cdots+1 & = & 1+1+\cdots+1+2+2+2+2 & & \\ = & 1+1+2+4+4+4 & = & 1+1+1+1+3+3+3+3 & & \\ = & 2+2+\cdots+2 & = & 1+1+1+1+2+2+\cdots+2 & & \\ = & 2+2+6+6 & = & 4+4+4+4 & = & 8+8 \end{array}$$

All of these cycle structures allow 0, 8, and 16. In particular, a submodule of dimension 8 cannot be ruled out by looking at cycle structures alone. So irreducibility cannot be seen from the cycle structures alone.

B_5: has cycles with structures $8+8+8+8$ and $2+10+10+10$. $8+8+8+8$ only allows for submodules of dimensions 0, 8, 16, 24, and 32 whereas $2+10+10+10$ only allows for submodules of dimensions 0, 2, 10, 12, 20, 22, 30, and 32. Thus, only 0 and 32 are allowed and so irreducibility follows.

B_6: Here the cycle structures all allow a submodule of dimension 24. So irreducibility cannot be deduced.

B_7: has cycles with structures $8+8+\cdots+8$, $2+14+14+\cdots+14$, and $4+4+20+20+\cdots+20$. These together rule out all possible submodules except those of dimensions 0 and 128 and so again irreducibility follows.

B_8: Cycle structures allow for a submodule of dimension 16. Fail.

B_9: Cycle structures allow for a submodule of dimension 144. Fail.

B_{10}: Cycle structures allow for a submodule of dimension 64. Fail.

B_{11}: Cycle structures allow for a submodule of dimension 288. Fail.

Conclusion:

Irreducibility of the minuscule representation of type B_n can be seen from cycle structures alone when $n = 2, 3, 5$, and 7. It cannot be seen from the cycle structures when $n = 4, 6, 8, 9, 10$, and 11. [Conjecture: It cannot be seen for all $n > 7$.]

Via random sampling we found compelling evidence that indicates that irreducibility cannot be seen from cycle structures alone for B_n with $n = 12, 13, \ldots, 23$. In fact, matters got worse and worse (with more and more dimensions allowed) and we increased the rank.

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