Reverse mathematics and marriage problems with unique solutions

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Abstract

We analyze the logical strength of theorems on marriage problems with unique solutions using the techniques of reverse mathematics, restricting our attention to problems in which each boy knows only finitely many girls. In general, these marriage theorems assert that if a marriage problem has a unique solution then there is a way to enumerate the boys so that for every m, the first m boys know exactly m girls. The strength of each theorem depends on whether the underlying marriage problem is finite, infinite, or bounded.

Our goal is to analyze the logical strength of some marriage theorems using the framework of reverse mathematics. The subsystems of second order arithmetic used here are RCA_0, which includes comprehension for recursive (also called computable) sets of natural numbers, WKL_0, which appends a weak form of König's Lemma for trees, and ACA_0, which appends comprehension for arithmetically definable sets. Simpson's book [5] provides detailed axiomatizations of the subsystems and extensive development of the program of reverse mathematics.

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We use the standard anthropocentric terminology for marriage theorems. A *marriage problem* consists of a set B of boys, a set G of girls, and a relation $R \subset B \times G$ where $(b, g) \in R$ means "b knows g." A *solution* of the marriage problem is an injection $f : B \to G$ such that for all $b \in B$, $(b, f(b)) \in R$. Informally, f assigns a spouse to each boy, chosen from among his acquaintances and avoiding polygamy. In general, marriage theorems provide necessary and sufficient conditions for the existence of solutions or, in our case, for the existence of unique solutions. These sorts of results are often expressed using other terminology such as transversals, systems of distinct representatives (SDRs), and matchings in bipartite graphs.

As a notational convenience, we use some set theoretic notation as abbreviations for more complicated formulas of second order arithmetic. If $b \in B$, we write $G(b)$ for $\{g \in G \mid (b, g) \in R\}$. In the marriage problems for this paper each boy knows at most finitely many girls, so for each b, RCA_0 can prove the existence of $G(b)$. Although $G(b)$ looks like function notation, it is not. In general, RCA_0 can prove the existence of a function uniformly mapping each boy to (the integer code for) the finite set $G(b)$ if and only if the marriage theorem is *bounded*, as defined after Theorem 2. We further abuse this notation by using formulas like $g \in G(B_0)$ to abbreviate $\exists b \in B_0((b, g) \in R)$. In settings that address more than one marriage problem, we write $G_M(B_0)$ to denote girls known by boys in B_0 in the marriage problem M. Cardinality notation like $|X| \leq |Y|$ abbreviates the assertion that there is an injection from X into Y. The formula $|X| < |Y|$ abbreviates the conjunction of $|X| \leq |Y|$ and $|Y| \not\leq |X|$. For finite sets, RCA_0 can prove many familiar statements about cardinality, for example, if X is finite and $y \notin X$ then $|X| < |X \cup \{y\}|$.

Given a marriage problem $M = (B, G, R)$ with a solution f, for any $B_0 \subset B$ the restriction of f is an injection of B_0 into $G(B_0)$. Consequently, RCA_0 proves that if M has a solution then $|B_0| \leq |G(B_0)|$ for every $B_0 \subset B$. Philip Hall [4] proved the converse for finite marriage problems. The following theorem shows that Philip Hall's result can be formalized and proved in RCA_0 and appears as Theorem 2.1 of Hirst [2].

Theorem 1. (RCA_0) *If $M = (B, G, R)$ is a finite marriage problem satisfying $|B_0| \leq |G(B_0)|$ for every $B_0 \subset B$, then M has a solution.*

Marshall Hall, Jr. [3] extended Philip Hall's theorem to infinite marriage problems. The following theorem shows that his result is equivalent to ACA_0 and appears as Theorem 2.2 of Hirst [2]. Marriage problems in which boys are allowed to know infinitely many girls are considerably more complex and not considered in this paper.

Theorem 2. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If $M = (B, G, R)$ is a marriage problem such that each boy knows only finitely many girls and $|B_0| \le |G(B_0)|$ for every finite $B_0 \subset B$, then M has a solution.*

Suppose that $M = (B, G, R)$ is a marriage problem in which B and G are subsets of $\mathbb{N}$. We say that M is *bounded* if there is a function $h : B \to G$ such that for each $b \in B$, $G(b) \subseteq \{0, 1, \dots, h(b)\}$. The function h acts as a uniform bound on the girls that each boy knows, and also insures that each boy knows only finitely many girls. Given such a bounding function, recursive comprehension proves the existence of the function mapping each b to (the code for) the finite set $G(b)$. As illustrated by the following theorem, bounded marriage theorems are often weaker than their unbounded analogs. The following appears as Theorem 2.3 of Hirst [2].

Theorem 3. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. *If $M = (B, G, R)$ is a bounded marriage problem such that $|B_0| \le |G(B_0)|$ for every finite $B_0 \subset B$, then M has a solution.*

Our goal is to analyze theorems on necessary and sufficient conditions for marriage problems to have unique solutions. A marriage problem with a single boy has a unique solution if and only if he knows exactly one girl. The following lemma shows that any finite marriage problem with a unique solution must contain such a boy.

Lemma 4. (RCA_0) *If $M = (B, G, R)$ is a finite marriage problem with a unique solution f, then some boy knows exactly one girl.*

Proof. Suppose we have M and f as above with $|B| = n$. Note that $|G_M(B)| = n$, since if $|G_M(B)| > n$ we could construct a new solution to M using a girl not in the range of f, contradicting the uniqueness of f.

Let s be the smallest number such that there is a $B_0 \subset B$ with $|B_0| = |G_M(B_0)| = s$. We know such an s exists by the Σ^0_0 least element principle, a consequence of Σ^0_1 induction. If $s = 1$ then we have proved the lemma. Suppose by way of contradiction that $s > 1$ and choose $b_0 \in B_0$. Since $s > 1$, $|G_M(b_0)| > 1$, so we may choose $g_1 \in G_M(b_0)$ such that $f(b_0) \neq g_1$. Consider $M' = (B_0 - \{b_0\}, G_M(B_0) - \{g_1\}, R')$ where R' is the restriction of R to the sets of M'. We claim that M' has no solution. To see this, let h be a solution of M′ and note that $h \cup (b_0, g_1)$ is a matching of $(B_0, G_M(B_0))$ distinct from f. Since $|B_0| = |G_M(B_0)|$, f matches boys not in B_0 to girls not in $G_M(B_0)$, so we may define

$$f'(b) = \begin{cases} g_1 & \text{if } b = b_0 \\ h(b) & \text{if } b \in B_0 - \{b_0\} \\ f(b) & \text{otherwise.} \end{cases}$$

This f' is a solution of M differing from f at b_0, contradicting the uniqueness of f. Thus M' has no solution. Apply Theorem 1 and find a set of boys $B_1 \subset B_0 - \{b_0\}$ who know too few girls, that is, $|B_1| > |G_{M'}(B_1)|$. Since f is a solution of M, $|B_1| \leq |G_{M'}(B_1) \cup \{g_1\}|$ so $|B_1| = |G_{M'}(B_1) \cup \{g_1\}| = |G_M(B_1)|$. However, $|B_1| < |B_0|$, contradicting the minimality of s. Therefore $s > 1$ cannot hold, completing the proof of the lemma. $\square$

Now we can formulate a theorem on unique solutions to finite marriage problems. Clearly, if we can line up the boys $b_1, b_2, \ldots, b_n$ so that for each $m \leq n$ the first m boys know exactly m girls, then the marriage problem has a unique solution. This implication is provable in RCA_0, as is its extension to infinite marriage problems. The following theorem shows that the converse for finite problems is provable in RCA_0. As noted by Chang [1], the combinatorial statement in the theorem is implicit in the work of Marshall Hall, Jr. [3].

Theorem 5. (RCA_0) *If $M = (B, G, R)$ is a finite marriage problem with n boys and a unique solution, then there is an enumeration of the boys $\langle b_i \rangle_{i \leq n}$ such that $|G(b_1, \ldots, b_m)|$ for every $1 \leq m \leq n$.*

Proof. Suppose M is as above. Working in RCA_0, we will construct a sequence of initial segments of $\langle b_i \rangle_{i \leq n}$. Apply Lemma 4 and let b_1 be the first boy (in some enumeration of B) such that $|G(b_1)| = 1$. Suppose that $t < n$, $\langle b_i \rangle_{i \leq t}$ is defined, and $|G(b_1, \ldots, b_t)| = t$. Since M has a unique solution, so does $M' = (B - \{b_1, \ldots b_t\}, G - G(b_1, \ldots, b_t), R')$, where R' is the restriction of R to the sets of M'. Apply Lemma 4 and let b_{t+1} be the first boy not in $\{b_1, \ldots, b_t\}$ such that $|G_{M'}(b_{t+1})| = 1$, completing the definition of $\langle b_i \rangle_{i \leq t+1}$. The desired enumeration is the n^{th} initial segment. $\square$

In light of the comments preceding Theorem 5, it could be reformulated as a biconditional statement, giving a necessary and sufficient condition for the existence of unique solutions to finite marriage problems. The same reformulation could be carried out for Theorems 7 and 9 below. Now we will analyze a version of Theorem 5 in the infinite setting, using ACA_0 in its proof. Paralleling the proof of Theorem 5, we begin with a lemma.

Lemma 6. (ACA_0) *Suppose $M = (B, G, R)$ is a marriage problem such that every boy knows finitely many girls and M has a unique solution. For any $b \in B$ there is a finite set F such that $b \in F \subset B$ and $|G(F)| = |F|$.*

Proof. Suppose f is the unique solution of $M = (B, G, R)$. Let $b \in B$. If $|G(b)| = 1$, the set $F = \{b\}$ satisfies the conclusion of the lemma. If $|G(b)| > 1$, a more complicated construction is required.

Assume $|G(b)| > 1$ and let $g_0 = f(b)$ and $G(b) = \{g_0, g_1, \ldots, g_m\}$. Consider the marriage problem $M_1 = (B - \{b\}, G - \{g_1\}, R_1)$ where R_1 denotes the restriction of R to the sets of M_1. Given a solution f_1 of M_1, the function $f_1 \cup (b, g_1)$ would be a solution of M distinct from f. Thus M_1 has no solution. Using ACA_0, we may apply Theorem 2 and find a finite collection of boys $E_1 \subset B - \{b\}$ such that $|E_1| > |G_{M_1}(E_1)|$. Thus $|E_1| \geq |G_{M_1}(E_1) \cup \{g_1\}|$ and since $G_{M_1}(E_1) \cup \{g_1\} = G_M(E_1)$, we have $|E_1| \geq |G_M(E_1)|$. Since M has a solution, $|E_1| \leq |G_M(E_1)|$, so combining inequalities shows that $|E_1| = |G_M(E_1)|$. For each i with $1 < i \leq m$ and each $g_i \in G(b)$ search for a similar finite set, finding an $E_i \subset B$ with $|G(E_i)| = |E_i|$ and $g_i \in G(E_i)$. Since each E_i is a finite set with an integer code, recursive comprehension suffices to prove the existence of the sequence $E_1, E_2, \ldots, E_m$ and the union $F = \{b\} \cup_{i \leq m} E_i$. Since each E_i is finite, F is finite and $b \in F \subset B$.

To complete the proof, we need only show that $|F| = |G(F)|$. Suppose by way of contradiction that $|G(F)| > |F|$. In this case, since f maps F into but not onto $G(F)$, we can choose $g \in G(F)$ such that for every $c \in F$, $f(c) \neq g$. Since $f(b) = g_0$, $g \neq g_0$. If $g \in G(b)$, then for some $i \geq 1$ we have $g = g_i$ and $g \in G(E_i)$. Since $G(F) = G(b) \cup \bigcup_{i \leq m} G(E_i)$, we may fix an i such that $g \in G(E_i)$. Since $|E_i| = |G(E_i)|$, f must map E_i onto $G(E_i)$, so for some $c \in E_i$, $f(c) = g$. This contradicts our choice of g, showing that $|G(F)| \leq |F|$. Since f is an injection of F into $G(F)$, we must have $|F| = |G(F)|$. $\square$

Note that the only use of ACA_0 in the preceding proof is the application of Theorem 2. This will be useful in adapting our results to the bounded marriage theorem setting.

Now we can analyze the extension of Theorem 5 to infinite marriage problems. Like the result of Marshall Hall, Jr. analyzed in Theorem 2, this statement is equivalent to ACA_0.

Theorem 7. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *Suppose $M = (B, G, R)$ is a marriage problem such that every boy knows finitely many girls. If M has a unique solution then there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that $|G(b_1, \ldots, b_n)| = n$ for every $n \geq 1$.*

Proof. To prove that (1) implies (2), we will work in RCA_0, making each application of ACA_0 explicit. Let $M = (B, G, R)$ be a marriage problem as described in (2). Let $\langle b'_i \rangle_{i \geq 1}$ be an arbitrary initial enumeration of B. Search for a finite set $F_1 \subset B$ such that $b'_1 \in F_1$ and $|G_M(F_1)| = |F_1|$. Define $n_1 = |F_1|$. By Lemma 6, this search must succeed. Note that ACA_0 is used here in the application of Lemma 6 and to determine the value of $|G_M(F_1)|$. We claim that f restricted to F_1 is a unique solution of $(F_1, G(F_1), R)$. To see this, suppose f' is a solution of $(F_1, G(F_1), R)$ differing from f. Since f is injective and maps F_1 onto $G(F_1)$, f must map $B - F_1$ into $G - G(F_1)$, Thus the extension of f' defined by

$$f'(b) = \begin{cases} f'(b) & \text{if } \ b \in F_1 \\ f(b) & \text{if } b \in B - F_1 \end{cases}$$

is a solution of M differing from f, contradicting the uniqueness of f. Since $(F_1, G(F_1), R)$ has a unique solution, by Theorem 5, there is an enumeration of the boys $\langle b^1_1, b^1_2, \ldots, b^1_{n_1} \rangle$ of F_1 such that $|G(b^1_1, b^1_2, \ldots, b^1_m)| = m$ for every m with $1 \leq m \leq n_1$.

Suppose $F_1, \ldots, F_j$ are sequences that have been constructed so that for each $i \leq j$ and each $t \leq n_i$, $F_i = \langle b^i_1, b^i_2, \ldots, b^i_{n_i} \rangle$ and

$$|G(\{b^i_1, b^i_2, \ldots, b^i_t\} \cup \bigcup_{k<i} F_k)| = t + \sum_{k<i} n_k.$$

Note that f restricted to $B - \cup_{k \leq j} F_j$ is a unique solution of the marriage problem $M_{j+1} = (B - \cup_{k \leq j} F_k, G - \cup_{k \leq j} G(F_k), R)$. Let b' denote the first element (in our initial enumeration of B) appearing in $B - \cup_{k \leq j} F_k$. Search for a finite set $F_{j+1} \subset B - \cup_{k \leq j} F_k$ such that $b' \in F_{j+1}$ and $|G_{M_{j+1}}(F_{j+1})| = |F_{j+1}|$. Define $n_{j+1} = |F_{j+1}|$. As before, Lemma 6 insures that the search will succeed. ACA_0 is applied here in the use of Lemma 6 and in determining values of $|G_{M_{j+1}}(F_{j+1})|$. As before, f restricted to F_{j+1} is a unique solution of M_{j+1} restricted to F_{j+1}, so by Theorem 5 there is an enumeration of the boys $\langle b_1^{j+1}, b_2^{j+1}, \ldots, b_{n_{j+1}}^{j+1} \rangle$ in F_{j+1} such that $G_{M_{j+1}}(b_1^{j+1}, \ldots, b_t^{j+1})|$ for every t with $1 \leq t \leq n_{j+1}$. Consequently, for every $1 \leq t \leq n_{j+1}$,

$$|G(\{b_1^{j+1}, b_2^{j+1}, \ldots, b_t^{j+1}\} \cup \bigcup_{k \leq j} F_k)| = t + \sum_{k \leq j} n_k.$$

Given the existence of each finite sequence F_j, recursive comprehension suffices to prove the existence of the concatenation of the finite sequences $\langle \langle b_1^j, \ldots, b_{n_j}^j \rangle \mid j \geq 1 \rangle$, and this sequence satisfies the conclusion of item (2) in the statement of the theorem.

To prove that (2) implies (1), we will work in RCA_0 and assume (2). By Lemma III.1.3 of Simpson [5], it suffices to use (2) to prove the existence of the range of an arbitrary injection. Let $f : \mathbb{N} \to \mathbb{N}$ be an injection. Using recursive comprehension, construct the marriage problem $M = (B, G, R)$ with

- $B = \{c_n \mid n \in \mathbb{N}\} \cup \{d_n \mid n \in \mathbb{N}\}$,
- $G = \{g_n \mid n \in \mathbb{N}\} \cup \{r_n \mid n \in \mathbb{N}\}$,
- for every i, $(c_i, g_i) \in R$ and $(d_i, r_i) \in R$, and
- if $f(m) = n$ then $(c_n, r_m) \in R$.

Let $h : B \to G$ such that $h(d_i) = r_i$ and $h(c_i) = g_i$ for each $i \in \mathbb{N}$. Clearly, h is injective and a solution to M. Note that any solution must match each d_i with r_i, thus no c_i can be matched to a r_i and so every c_i must be matched with g_i. Hence, h is a unique solution to M.

Apply item (2) and let $\langle b_i \rangle_{i \geq 1}$ be an enumeration of B such that for every $n \geq 1$ we have $|G(b_1, \ldots, b_n)| = n$. Suppose $f(j) = k$. Then $(c_k, r_j) \in R$ and $G(c_k) = \{g_k, r_j\}$. Since $c_k \in B$, for some n we have $c_k = b_n$. If $d_j \notin \{b_1, \ldots, b_{n-1}\}$ then for each $i \leq n - 1$, $G(b_i) \cap G(c_k) = \emptyset$. In this case,

$$|G(b_1, \ldots, b_n)| = |G(b_1, \ldots, b_{n-1})| + |G(c_k)| = (n - 1) + 2 = n + 1,$$

contradicting $|G(b_1, \ldots, b_n)| = n$. Summarizing, whenever $f(j) = k$, d_j must appear before c_k in the enumeration of the boys. Thus k is in the range of f if and only if for some b appearing before c_k in the enumeration, $b = d_j$ and $f(j) = k$. Since we need only check finitely many values of f to see if k is in the range, recursive comprehension proves the existence of the range of f, completing the proof of the theorem. $\square$

Like Theorem 5, the preceding theorem continues to hold if the implication in item (2) is changed to a biconditional. While such a formulation provides a complete characterization of the marriage problems with unique solutions, it weakens the statement of the reversal.

As noted in the introduction, bounded marriage problems are often weaker than their unbounded versions. This is also true for the following bounded analogs of Lemma 6 and Theorem 7, as shown by the next two results.

Lemma 8. (WKL_0) *Suppose $M = (B, G, R)$ is a bounded marriage problem and M has a unique solution. For any $b \in B$ there is a finite set F such that $b \in F \subset B$ and $|G(F)| = |F|$.*

Proof. Proceed exactly as in the proof of Lemma 6, replacing each application of Theorem 2 with an application of Theorem 3. $\square$

Theorem 9. (RCA_0) *The following are equivalent:*

1. WKL_0.

2. *Suppose $M = (B, G, R)$ is a bounded marriage problem. If M has a unique solution then there is an enumeration of the boys $\langle b_i \rangle_{i \geq 1}$ such that $|G(b_1, \ldots, b_n)| = n$ for every $n \geq 1$.*

Proof. To prove that (1) implies (2), repeat the corresponding portion of the proof of Theorem 7, replacing uses of ACA_0 with uses of WKL_0 as follows. First, substitute Lemma 8 for Lemma 6 everywhere. Note that the bounding function for M also acts as a bounding function for any marriage problem created by deleting sets of boys and girls from M. Consequently, we may use the bounding function and recursive comprehension to compute $G(F)$ for each finite set F whenever required.

The reversal requires a completely new argument. We will use (2) to prove that any binary tree (with nodes labeled 0 or 1) with no infinite paths is finite. Toward this end, suppose T is a binary tree with no infinite paths.

As in section III.7 of Simpson [5], we can identify each node of T with a binary sequence, $\sigma \in 2^{<\mathbb{N}}$. Construct the marriage problem $M = (B, G, R)$ by letting $B = \{b_\sigma \mid \sigma \in T\}$, $G = \{g_\sigma \mid \sigma \in T\}$, and

$$R = \{(b_\sigma, g_\sigma) \mid \sigma \in T\} \cup \{(b_\sigma, g_{\sigma^\frown i}) \mid \sigma \in T \wedge \sigma^\frown i \in T\}.$$

Note that since $G(b_\sigma) = \{g_\sigma, g_{\sigma^\frown 0}, g_{\sigma^\frown 1}\} \cap G$ for all σ, M is a bounded marriage problem.

Define $f : B \to G$ by $f(b_\sigma) = g_\sigma$. We claim that f is a unique solution of M. To verify this let f_1 be a second solution of M where $f_1(b_\sigma) = g_{\sigma^\frown i}$ for some $i \in \{0, 1\}$. Fix such a σ and i and let $\sigma_0 = \sigma$. Given σ_n let $\sigma_{n+1} = \tau$ where $f_1(b_{\sigma_n}) = g_\tau$. Since f_1 is injective, an easy induction argument shows that σ_{n+1} must always be an extension of σ_n. Hence, $\langle \sigma_n \mid n \in \mathbb{N}\rangle$ forms an infinite path through T, yielding a contradiction. Thus f is unique.

Since M has a unique solution, we can apply item (2) to M and enumerate the boys $\langle b_i \rangle_{i \geq 1}$ so that $|G(b_1, \ldots, b_n)| = n$ for all n. Recursive comprehension proves the existence of translation functions between the two types of subscripting on the boys. Let $r : T \to \mathbb{N}$ be the bijection defined by $r(\sigma) = n$ if and only if $b_n = b_\sigma$.

We claim that each boy appears in the enumeration after all of his proper successors in the tree. Using $\sigma \prec \tau$ to denote that σ is a proper initial segment of τ, our claim becomes: if $\sigma \prec \tau \in T$, then $r(\sigma) > r(\tau)$. To prove this, suppose by way of contradiction that for some $\sigma \prec \tau \in T$, $r(\sigma) < r(\tau)$. By the Σ^0_0 least element principle (a consequence of Σ^0_1 induction) we can fix a shortest sequence τ such that $r(\sigma) < r(\tau)$ for some $\sigma \prec \tau$. Because τ is shortest, there is no α such that $r(\alpha) > r(\tau)$ and $\sigma \prec \alpha \prec \tau$. Thus we may assume that τ is an immediate successor of σ. Summarizing, we have $r(\sigma) < r(\tau)$ and $\tau = \sigma^\frown i$ for some $i \in \{0, 1\}$. Let $B' = \{b_1, b_2, \ldots, b_{r(\sigma)}\}$. Using the node-based indices for the boys, we can write $B' = \{b_\alpha \mid r(\alpha) \leq r(\sigma)\}$. Note that $r(\tau) > r(\sigma)$, so $b_\tau \notin B'$. By the definition of M, $(b_\sigma, g_\tau) \in R$, so $g_\tau \in G(B')$. Also, for every $b_\alpha \in B'$, $(b_\alpha, g_\alpha) \in R$, so $g_\alpha \in G(B')$. Thus $|G(B')| > |B'|$. However, B' is an initial segment of the enumeration of B provided by the application of (2), so $|G(B')| = |B'|$. This contradiction completes the proof that each boy appears in the enumeration after all of his proper successors in the tree.

The empty sequence $\langle\ \rangle$ is in T, so for some n, $b_{\langle\ \rangle} = b_n$ Since b_n appears after every boy corresponding to a nonempty node of T, we know T is finite, completing the proof of the reversal and the theorem. $\square$

At this time, we have been unable to determine the exact strength of some of the lemmas in the preceding material. For example, although we know that Lemma 6 is provable in ACA_0, we do not know if it can be proved in a weaker subsystem. Consider the following formulation of an infinite version of Lemma 4: If M is a marriage problem in which each boy knows only finitely many girls and M has a unique solution, then some boy knows exactly one girl. Lemma 6 and Lemma 4 give a proof in ACA_0, but the following theorem shows that at most WKL_0 is required.

Theorem 10. (WKL_0) *Suppose M is a marriage problem in which every boy knows at least two girls and at most finitely many girls. If M has a solution, then M has at least two solutions.*

Proof. Assume WKL_0. Let $M = (B, G, R)$ be a marriage problem with solution $f : B \to G$ and suppose that every boy knows at least two girls. Define a function $h_0 : B \to G$ by letting $h_0(b)$ be the first girl other than $f(b)$ that b knows. Formally, $h_0(b) = \mu g((b, g) \in R \wedge f(b) \neq g)$. Define $h_1 : B \to G$ by $h_1(b) = \max\{h_0(b), f(b)\}$ and let $R' = \{(b, g) \mid f(b) = g \vee h_0(b) = g\}$. Recursive comprehension proves the existence of h_0, h_1, and R'. The society $M' = (B, G, R')$ is bounded by h_1 and has f as a solution. Every boy in M' knows exactly two girls. By Theorem 9, if f is a unique solution of M', then some boy in B knows exactly one girl, contradicting the construction of M'. Consequently, M' has at least two solutions. Since every solution of M' is also a solution M, M also has at least two solutions. $\square$

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