Math 2410-004/013 **Name:** Solutions **Due:** 3/23/18

Homework #5: Appendix B

In class, we saw that matrix multiplication was not commutative. This means, for example that if A and B are both 2×2 matrices, then it may be the case that AB and BA are different matrices, that is $AB \neq BA$. This however, begs the question of which properties of *real multiplication* (i.e. multiplication of real numbers) do generalize to matrix multiplication. Real multiplication has a multiplicative identity, the number 1 ($x \cdot 1 = 1 \cdot x = x$) and we have seen in class that matrix multiplication has a multiplicative identity as well, namely the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If A and I are of the proper size we have $AI = A = IA$. (Can you determine the additive identities for real and matrix multiplication?)
In the following two questions, we will generalize two more properties of real multiplication to matrix multiplication.

1. Here we will show that the *distributive property* ($a(x + y) = ax + ay$ for real numbers a, x and y) holds for matrix multiplication. To simplify the computations we restrict our attention to 2×2 matrices and 2×1 column vectors. Specifically, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Show that matrix multiplication distributes over matrix addition by verifying that

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$

Solution: As real multiplication distributes, we have the following

$$\begin{aligned} A(\mathbf{x} + \mathbf{y}) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} a(x_1 + y_1) + b(x_2 + y_2) \\ c(x_1 + y_1) + d(x_2 + y_2) \end{pmatrix} \\ &= \begin{pmatrix} ax_1 + ay_1 + bx_2 + by_2 \\ cx_1 + cy_1 + dx_2 + dy_2 \end{pmatrix} \\ &= \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} + \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\mathbf{x} + A\mathbf{y} \end{aligned}$$

Distribution can be shown in full generality: that is if A, B and C have the necessary products defined then

$$A(B+C) = AB + AC \text{ and } (B+C)A = BA + CA.$$

You do not need to do this.
For two square matrices A and B, we can now add them $(A+B)$, subtract them $(A - B = A + (-1)B)$ and multiply them (AB **or** BA). Is there any way to generalize division to matrices? Though we cannot straight-forwardly define "matrix division" per se, we can define a suitable analog. In the real numbers, dividing by a number a is equivalent to multiplying by the number $1/a$ or a^{-1} where a^{-1} is the unique number such that

$$a \cdot a^{-1} = \frac{a}{a} = 1.$$

Indeed $4 \div 2 = 4 \cdot 2^{-1} = 4 \cdot (1/2) = 2$ and $2 \div 2 = 2 \cdot 2^{-1} = 2 \cdot (1/2) = 1$. For a given number a, we call the number a^{-1} it's multiplicative inverse for it essentially cancels the effect of multiplying by a.
Thus our question of generalizing division to matrix multiplication becomes "for a given matrix A, can we find another matrix B so that $AB = BA = I$?" For certain matrices, we can, and we call the matrix B the *multiplicative inverse* of A. The next two questions deal with when we can, and how we, find such an inverse.

2. Let

$$A = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}, B = \begin{pmatrix} -4 & 7 \\ 1 & -2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix}.$$

Determine which of B and C is the multiplicative inverse of A and label it A^{-1}.

Hint: $AA^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution: If B is the multiplicative inverse of A, then it has to be the case that $AB = I$. Notice

$$AB = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -4 & 7 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 8-7 & 14-14 \\ -4+4 & 7-8 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

As $AB \neq I$, we must have that $AC = I$. Checking that product,

$$AC = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8-7 & 14-14 \\ 4-4 & -7+8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see indeed that $AC = I$ and thus $C = A^{-1}$.

Now that we see matrices have inverses, how do we find the inverse? Well, when trying to determine an $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ for some matrix A and vector b, we used row reduction on the augmented matrix $(A \mid \mathbf{b})$. As it turns out, this approach will also work for finding matrix inverses. Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

if it has an inverse matrix A^{-1} such that $AA^{-1} = I$, we can determine A^{-1} by row reducing the augmented matrix $(A \mid I)$. (For an example see App-21 in the text.)

3. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(a) Write the augmented matrix $(A \mid I)$ where I is the 3×3 identity matrix.

Solution:

$$\left(\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

(b) Row reduce your solution to part (a). You should obtain a matrix of the form $(I \mid B)$ for some 3×3 B.

Solution:

$$\left(\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2+(-1)R_1} \left(\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+(-1)R_2} \left(\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1+R_3} \xrightarrow{(-1)R_3} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_{23}} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

(c) Verify that the matrix B you found in part (b) is the inverse of A. Label it A^{-1}.

Solution: From part (b) we found

$$B = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

To see that B is indeed A^{-1}, we consider the product AB:

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2-1+0 & -1+1+0 & 1-1+0 \\ 2-1-1 & 1-1+1 & 1-1+0 \\ 0+1-1 & 0-1+1 & 0+1+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I. \end{aligned}$$

As $AB = I$, we have that $B = A^{-1}$.

Having the inverse of a coefficient matrix makes it quite simple to solve linear systems with that coefficient matrix. For example, the system

$$\begin{aligned} ax + by &= b_1 \\ cx + dy &= b_2 \end{aligned}$$

can be written in matrix notation as $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
If A has an inverse, then we can find the values of x and y as follows

$$A\mathbf{x} = \mathbf{b} \implies A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b} \implies \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

4. Consider the system

$$\begin{aligned} x + 2y &= b_1 \\ 3x + 5y &= b_2. \end{aligned}$$

(a) Rewrite this system as a matrix equation.

Solution: If $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, we can rewrite the system as the matrix equation

$$A\mathbf{x} = \mathbf{b} \text{ or } = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

(b) Find the inverse of the coefficient matrix.

Solution: We row reduce the matrix $(A|I)$:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \xrightarrow{R_2+(-3)R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right) \xrightarrow[-R_2]{R_1+2R_2} \left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

Thus, the inverse of A is

$$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

(c) Use this to solve the three systems in which

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

without row reduction.

Solution: Let

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{b}_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Note if $A\mathbf{x} = \mathbf{b}$, then $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$ which implies that $\mathbf{x} = A^{-1}\mathbf{b}$. Thus, if x_i is the solution to $A\mathbf{x} = \mathbf{b}_i$ for $i = 1, 2$ or 3 we have

$$\begin{aligned} \mathbf{x}_1 &= A^{-1}\mathbf{b}_1 = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \\ \mathbf{x}_2 &= A^{-1}\mathbf{b}_2 = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \mathbf{x}_3 &= A^{-1}\mathbf{b}_3 = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \end{aligned}$$

Not all matrices have inverses, we will not investigate why that is here, but we will state the following fact.

Fact: If A a matrix such that $\det A = 0$, the A has no inverse and is called *noninvertible* or *singular*.

As we saw in the previous problem, if a coefficient matrix has an inverse, than the corresponding system has a unique solution. When a matrix is non-invertible, then a system with that coefficient matrix may have 0, 1, or infinitely many solutions.
We'll investigate this next.

5. Consider the system

$$\begin{aligned} x + 3y - 2z &= -7 \\ 4x + y + 3z &= 5 \\ 2x - 5y + 7z &= 19 \end{aligned}$$

(a) Rewrite this system as a matrix equation. Give the associated augmented matrix for this system.

Solution: If $A = \begin{pmatrix} 1 & 3 & -2 \\ 4 & 1 & 3 \\ 2 & -5 & 7 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -7 \\ 5 \\ 19 \end{pmatrix}$ then the matrix equation which encodes this system is

$$A\mathbf{x} = \mathbf{b} \text{ or } \begin{pmatrix} 1 & 3 & -2 \\ 4 & 1 & 3 \\ 2 & -5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ 19 \end{pmatrix}.$$

The associated augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right)$$

(b) Row reduce the augmented matrix given above and translate the resulting matrix into a system of equations.
Hint: You should obtain only two equations.

Solution: Row reducing yields

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right) \xrightarrow[R_3+(-2)R_1]{R_2+(-4)R_1} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right) \xrightarrow{R_3+(-1)R_2} \xrightarrow{\frac{-1}{11}R_2} \xrightarrow{R_1+(-3)R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which translating back into a system of equations gives

$$\begin{aligned} x + z &= 2 \\ y - z &= -3 \end{aligned}$$

(c) Let k be any real number and set $z = k$. Give the solution to the original system in terms of the relationship between x, y and z with k.

Solution: If $z = k$, then $x + k = 2$ and $y - k = -3$. Hence, for any choice of k, we have a solution to the system given by

$$\begin{aligned} x &= 2 - k \\ y &= -3 - k \\ z &= k \end{aligned}$$

When finding the eigenvalues of a matrix A, we find λ such that the matrix $A - \lambda I$ has determinant zero. That is, we choose λ such that $A - \lambda I$ does not have an inverse. This ensures that the system of equations defined by

$$A\mathbf{x} = \lambda\mathbf{x} \implies (A - \lambda I)\mathbf{x} = \mathbf{0}$$

has infinitely many nonzero solutions. We finish this homework by showing that if λ is an eigenvalue of A, then λ must have infinitely many eigenvectors.

6. Let A be a matrix with eigenvalue λ for which $\mathbf{v}$ is an eigenvector. Prove that for any constant k, $k\mathbf{v}$ is also an eigenvector of A with eigenvalue λ. This shows that λ has infinitely many distinct eigenvalues.

Solution: To show that $k\mathbf{v}$ is an eigenvector, we simply need show $A(k\mathbf{v}) = \lambda(k\mathbf{v})$. To that end, note

$$A(k\mathbf{v}) = Ak\mathbf{v} = kA\mathbf{v} = k(A\mathbf{v}) = k(\lambda\mathbf{v}) = \lambda k\mathbf{v} = \lambda(k\mathbf{v}),$$

as desired. This completes the proof.