Math 2410-004/013 **Name:** Solutions **Due:** N/A

Homework #4: Sections 2.6, 2.3, 2.5

1. Use Euler's method with step size $h = 0.1$ to approximate $y(1.2)$, where $y(x)$ is the solution to the IVP

$$\frac{dy}{dx} = 1 + x\sqrt{y}, \quad y(1) = 9.$$

Solution: Here $\Delta x = 0.1$ and $f(x, y) = 1+x\sqrt{y}$. Recalling that Euler's method obtains (x_{k+1}, y_{k+1}) from (x_k, y_k) by way of

$$\begin{aligned} x_{k+1} &= x_k + \Delta x \\ y_{k+1} &= y_k + \Delta x f(x_k, y_k) \end{aligned}$$

we obtain the following table:

k	x_k	y_k	$f(x_k, y_k)$
0	1	9	$1 + 1\sqrt{9} = 4$
1	1.1	$9 + 0.1(4) = 9.4$	$1 + (1.1)\sqrt{9.4} \approx 4.373$
2	1.2	$9.4 + (0.1)(4.373) = 9.8373$	

Hence, we have the following approximation of the solution to the IVP at $x = 1.2$:

$$y(1.2) \approx 9.8373.$$

2. Consider the piecewise defined linear differential equation

$$\frac{dy}{dx} + 2xy = f(x), \text{ where } f(x) = \begin{cases} x & \text{if } 0 \leqslant x < 1 \\ 0 & \text{if } x \geqslant 1 \end{cases}$$

Here we will find a continuous function $y(x)$ which satisfies this differential equation as well as the initial condition that $y(0) = 2$.

(a) Solve the IVP

$$\frac{dy}{dx} + 2xy = f(x), \qquad y(0) = 2$$

on the interval $[0, 1)$. Call this solution $y_1(x)$.

Solution: As we are only considering x in the interval $[0, 1)$, this problem reduces to solving the IVP

$$\frac{dy}{dx} + 2xy = x, \qquad y(0) = 2.$$

First, noting that $2x$ and x are continuous on $[0, 1)$ allows us to conclude that the method of integrating factors will yield the general solution to the DE on this interval. The integrating factor in this case is

$$\mu(x) = e^{\int 2x\, dx} = e^{x^2}.$$

Hence the general solution is

$$y(x) = e^{-x^2} \int x e^{x^2}\, dx + c e^{-x^2} = \frac{1}{2} + c e^{-x^2}.$$

To satisfy the initial condition, we require that

$$y(0) = \frac{1}{2} + c e^{0} = \frac{1}{2} + c = 2.$$

So $c = 3/2$. Hence, we arrive at the particular solution to the IVP

$$y_1(x) = \frac{1 + 3e^{-x^2}}{2}$$

(b) Find a family of solutions to the differential equation

$$\frac{dy}{dx} + 2xy = f(x)$$

on the interval $(1, \infty)$. Use c for your constant of integration and call this family of solutions $y_2(x)$.

Solution: Here, like above, reduces to solving in general the DE

$$\frac{dy}{dx} + 2xy = 0.$$

Again, we may use the integrating factor $\mu(x) = e^{x^2}$ as $2x$ and 0 are continuous on the given interval $(1, \infty)$. Thus, the family of solutions required is

$$y_2(x) = e^{-x^2} \int e^{x^2} \cdot 0\, dx + c e^{-x^2} = c e^{-x^2}$$

(c) Using your answers from the previous two parts, define the piecewise function

$$y(x) = \begin{cases} y_1(x) & \text{if } 0 \leqslant x < 1 \\ y_2(x) & \text{if } x \geqslant 1 \end{cases}$$

Find a value of c such that this function is continuous.

Solution: Following the prompt we have

$$y(x) = \begin{cases} \dfrac{1 + 3e^{-x^2}}{2} & \text{if } 0 \leqslant x < 1 \\ c e^{-x^2} & \text{if } x \geqslant 1 \end{cases}$$

This will only be continuous if c is chose so that $y_1(1) = y_2(1)$. Here

$$y_1(1) = \frac{1 + 3e^{-1}}{2} = c e^{-1} = y_2(1)$$

if and only if c is such that

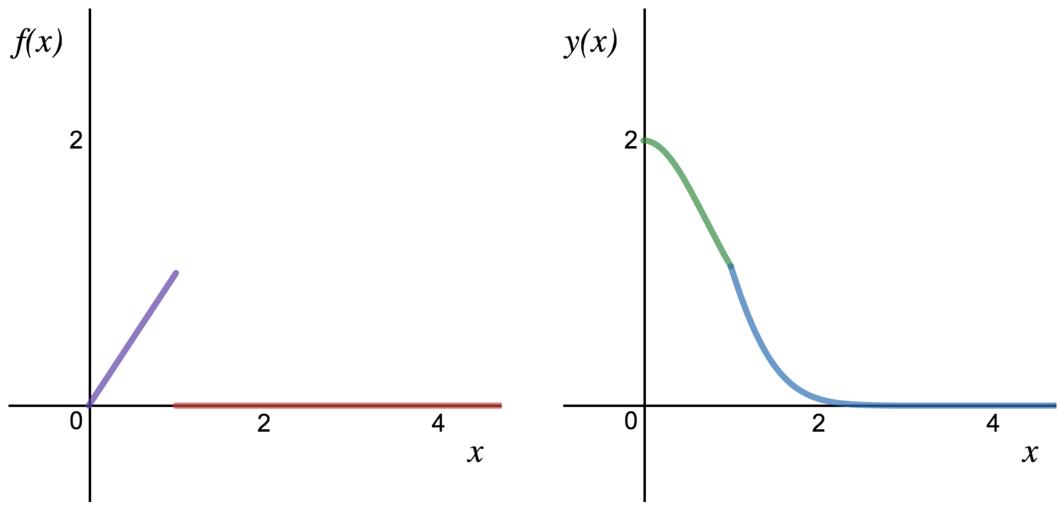
$$1 + 3e^{-1} = 2ce^{-1} \iff e = 2c - 3 \iff c = \frac{e+3}{2}.$$

Thus the continous function which satisfies the differential equation on the interval $[0, \infty)$ is

$$y(x) = \begin{cases} \dfrac{1 + 3e^{-x^2}}{2} & \text{if } 0 \leqslant x < 1 \\ \dfrac{e + 3}{2e^{x^2}} & \text{if } x \geqslant 1 \end{cases}$$

(d) Discuss why it is not fit to formally refer to $y(x)$ as a *solution* on $[0, \infty)$ but instead, simply as a continuous function which satisfies the differential equation on $[0, \infty)$.
It may help to plot $f(x)$ and $y(x)$ over the interval $(0, 5)$.

Solution: Below are the plots of $f(x)$ and $y(x)$ (color has been used to differentiate between the constituents of the piecewise functions).



A *solution* is to first-order differential equation is a *continuously differentiable* function defined on an interval I.
The interval in question here is $[0, \infty)$ which, due to the cusp at $x = 1$, we see $y(x)$ is not continuously differentiable on. Hence, $y(x)$ cannot technically be considered a solution to the DE.

3. By using an appropriate substitution, solve the following differential equation

$$\frac{dy}{dx} - y = e^x y^2.$$

Solution: This differential equation is of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

with $n = 2$. Hence it is a Bernoulli equation and using the substitution $u = y^{1-2} = y^{-1}$ will transform this equation into a linear DE.
Recall that if $y = g(x, u(x))$ then we use the multivariate chain rule to determine the relationship between y' and u'. Specifically

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx}.$$

Here since $y = u^{-1}$ we have

$$\frac{dy}{dx} = 0 \cdot 1 + (-u^{-2})\frac{du}{dx} = \frac{-1}{u^2}\frac{du}{dx}.$$

Substituting into the given differential equation yields

$$\frac{-1}{u^2}\frac{du}{dx} - \frac{1}{u} = \frac{e^x}{u^2}.$$

Multiplying through by $-u^2$ yields the linear equation

$$\frac{du}{dx} + u = e^x,$$

which on $(-\infty, \infty)$, has integrating factor $\mu(x) = e^x$. Thus, the solution to this equation (in u) is

$$u(x) = \frac{1}{e^x}\int e^{2x}\,dx + \frac{c}{e^x} = \frac{e^{2x} + c}{2e^x}$$

with c an arbitrary constant. From this, noting $y = u^{-1}$ we obtain the general solution to the original differential equation (in y)

$$y(x) = \frac{2e^x}{e^{2x} + c}$$

on $(-\infty, \infty)$.