Math 2410-004/013 **Name:** Solutions **Due:** 2/2/18

Homework #2: Sections 1.2, 1.3, 2.2

1. (a) Verify that $3x^2 - y^2 = c$ is a one-parameter family of implicit solutions of the differential equation $y\dfrac{dy}{dx} = 3x$.

Solution: As these solutions are implicit, we need only verify that they satisfy the differential equation. Determining upon which intervals the functions defined from this implicit family are solutions can be left until we consider specific explicit solutions.
Using implicit differentiation we see that

$$\frac{d}{dx}(3x^2 - y^2) = \frac{d}{dx}(c) \Longrightarrow 6x - 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = \frac{3x}{y}.$$

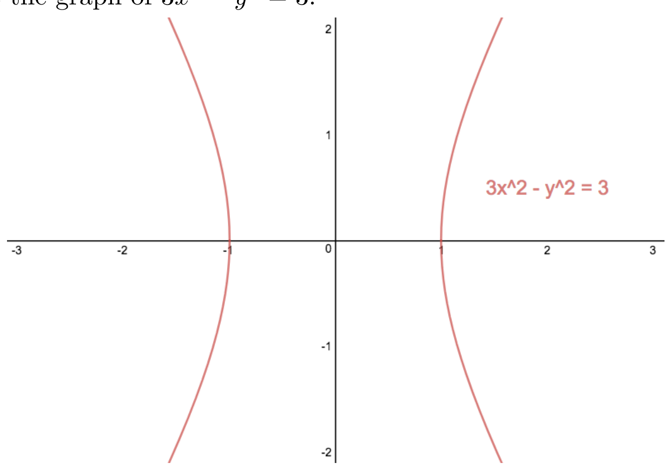
Now, noticing that

$$y\frac{dy}{dx} = y \cdot \frac{3x}{y} = 3x,$$

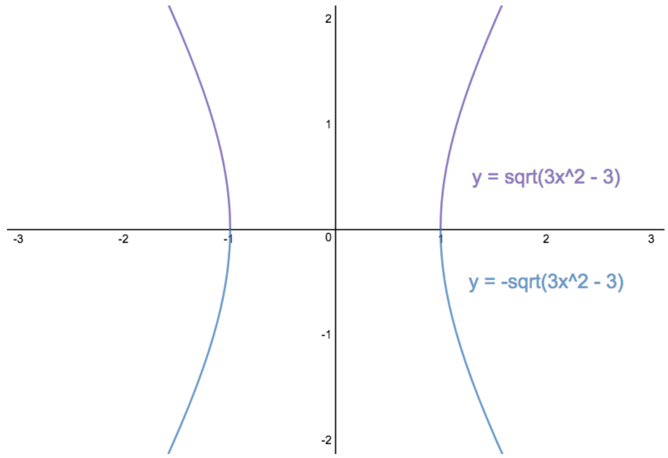
we see that is indeed a family of implicit solutions to the differential equation $y\dfrac{dy}{dx} = 3x$.

(b) Sketch the graph of the particular solution $3x^2 - y^2 = 3$. Find all explicit solutions $y = \varphi(x)$ of the DE in part (a) defined by this relation (whose interval of definition is as large as possible.) Specify the interval of definition I for each such solution.

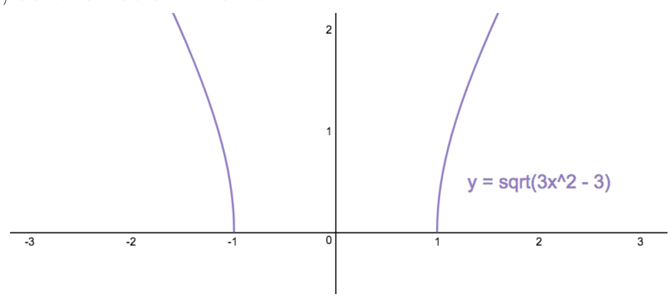
Solution: Here is the graph of $3x^2 - y^2 = 3$.



To determine explicit solutions from this implicit family we solve the equation $3x^2 - y^2 = 3$ for y to obtain $y = \pm\sqrt{3x^2 - 3}$. Note that this yields the two functions $y = \sqrt{3x^2 - 3}$ and $y = -\sqrt{3x^2 - 3}$, plotted below.



However, neither of these functions are solutions on their entire domain as their domain, $(-\infty, -1] \cup [1, \infty)$, as it is not an interval.



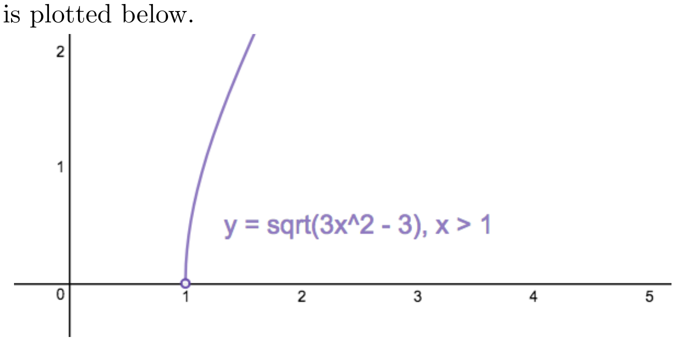
To turn these functions into solutions, we need to restrict them to an interval they are continuously differentiable on. This is nearly any interval contained in their domain. However, the derivatives of both functions

$$\frac{d}{dx}\left(\sqrt{3x^2-3}\right) = \frac{3x}{\sqrt{3x^2-3}} \quad \text{and} \quad \frac{d}{dx}\left(-\sqrt{3x^2-3}\right) = \frac{-3x}{\sqrt{3x^2-3}}$$

are discontinuous at $x = 1$ so any interval we restrict to must not contain $x = 1$.
With this in mind, we conclude that the four solutions we can define from $3x^2 - y^2 = 3$ are as follows

$$y = \sqrt{3x^2-3} \text{ defined on } (1, \infty)$$
$$y = \sqrt{3x^2-3} \text{ defined on } (-\infty, -1)$$
$$y = -\sqrt{3x^2-3} \text{ defined on } (1, \infty)$$
$$y = -\sqrt{3x^2-3} \text{ defined on } (-\infty, -1)$$

The first of these is plotted below.



(c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$. Which of the solutions you found in part (b) satisfy the IVP

$$y\frac{dy}{dx} = 3x \qquad y(-2) = 3.$$

Solution: Only two of the functions can take negative input, namely

$$y = \sqrt{3x^2 - 3} \text{ defined on } (-1, \infty) \text{ and } y = -\sqrt{3x^2 - 3} \text{ defined on } (-1, \infty).$$

The latter function always has negative output, so it must be the first function: $y = \sqrt{3x^2 - 3}$ defined on $(-1, \infty)$. Testing $y(-2)$ will indeed yield 3.

2. Suppose that the first-order differential equation $y' = f(x, y)$ possesses a one-parameter family of solutions and that $f(x, y)$ satisfies the hypotheses of the existence and uniqueness theorem on some rectangle R in the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R.

Solution: Because the uniqueness theorem applies to this differential equation near every possible initial condition (x_0, y_0), if $y_1(x)$ solves $y' = f(x, y)$ with $y_1(x_0) = y_0$ then if any other function, say $y_2(x)$ solves this IVP as well, it must be that $y_2 = y_1$ since y_1 is guaranteed to be unique.

Now, if two solution curves, say y_1 and y_2, intersect or are tangent at (x_0, y_0), that is, if

$$y_1(x_0) = y_0 = y_2(x_0)$$

then both of these functions solve the IVP

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Hence, by uniqueness they must be the same function, that is $y_1 = y_2$. In other words two solution curves cannot touch, for then, they would be the same solution curve.

3. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Pure water is pumped into the tank at a rate of 3 gallons every minute and when the solution is well stirred, it is then pumped out at the same rate.

(a) Determine a differential equation for the amount of salt $A(t)$ in the tank at time $t > 0$.

Solution: Proceeding as we did in lecture, we let $A(t)$ represent the amount of salt in pounds in the tank at time t, where t is time in minutes.

We know dA/dt is the difference of salt entering the tank and salt leaving the tank. No salt enters the tank, as only pure water is introduced. Salt does leave the tank however, at a rate of 3 gallons a minute in which each gallon contains $A/300$ pounds of salt. Establishing this as a differential equation renders

$$\frac{dA}{dt} = (\text{flow in}) - (\text{flow out}) = 0 - 3 \cdot \frac{A}{300}.$$

Thus, the desired IVP for the system is

$$\frac{dA}{dt} = \frac{-A}{100}, \quad A(0) = 50.$$

(b) Determine $A(0)$ and use this as an initial condition to construct an IVP using the differential equation found in part (a). Solve this IVP.

Solution: Clearly, we have $A(0) = 50$ as this is the initial amount of salt in the tank. Thus, the desired IVP for the system is

$$\frac{dA}{dt} = \frac{-A}{100}, \quad A(0) = 50.$$

Since the differential equation is separable, we may apply separation of variables to obtain

$$\int \frac{1}{A}\, dA = \int \frac{-1}{100}\, dt \ln|A| = \frac{-t}{100} + c \Longrightarrow A(t) = ke^{\frac{-t}{100}}, \quad k \neq 0.$$

(Note we are assuming $A \neq 0$ but this causes no issue as our solution will satisfy the initial condition $A(0) = 50 \neq 0$.) To determine k, we see that

$$A(0) = ke^0 = 50 \text{ if } k = 50.$$

Thus, the solution to the IVP is $A(t) = 50e^{\frac{-t}{100}}$.

(c) At what time will the concentration of salt in the solution be half of it's initial value?

Solution: Since the tank started with 50 pounds of salt, half of its initial concentration will translate to 25 pounds of salt in the tank. Hence, we ask for what t does $A(t) = 25$? Quickly, we find that

$$25 = A(t) = 50e^{\frac{-t}{100}} \iff -100\ln(1/2) = t.$$

Roughly, after $t \approx 69.31$ minutes.

(d) **Bonus:** (2 points) Will the tank ever be free of salt? Discuss your answer in terms of limits.

Solution: According to our model, no. The function $A(t) = 50e^{\frac{-t}{100}}$ is always positive. After 10 hours, or $t = 600$ minutes, there is less than 0.12 pounds of salt in the tank. After 20 hours there is less than 0.0005 pounds of salt in the tank, but the model states that there will never, at any finite time, be pure water in the tank.

What does the model predict however? That in the long run, the tank will become pure. This fits our intuition as the amount of salt should indefinitely decrease. In formal terms, we mean that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} 50e^{\frac{-t}{100}} = 0.$$

4. Solve the following IVP:

$$t^2\frac{dx}{dt} = x - tx, \quad x(-1) = -1.$$

Solution: Rewriting this differential equation in the form

$$\frac{dx}{dt} = x\left(\frac{1-t}{t^2}\right)$$

shows that this is a separable DE. Hence, we apply separation of variables and obtain

$$\int \frac{1}{x}\, dx = \int \frac{1-t}{t^2}\, dt \Longrightarrow \ln|x| = \int \frac{1}{t^2} - \frac{1}{t}\, dt \Longrightarrow \ln|x| = -\frac{1}{t} - \ln|t| + c.$$

Solving for x, we have that

$$\begin{aligned} x(t) &= ke^{\frac{-1}{t} - \ln|t|} \\ &= ke^{\frac{-1}{t}} e^{-\ln|t|} \\ &= ke^{\frac{-1}{t}} \frac{1}{t} = \frac{ke^{-1/t}}{t}. \end{aligned}$$

Here again we have assumed that $x \neq 0$, but this does not lose any information pertinent to this IVP as $x(-1) = -1 \neq 0$. Applying the initial condition, we find k:

$$x(-1) = \frac{ke^{-1/(-1)}}{-1} = -ke = 1 \Longrightarrow k = \frac{-1}{e}.$$

Hence, the solution to the IVP is

$$x(t) = \frac{e^{-1-\frac{1}{t}}}{t}.$$