Math 2410-004/013 **Name:** Solutions **Due:** 4/27/18

Homework #7: The Laplace transform

In this homework you will verify a few properties of the Laplace transform. Recall the definition is as follows

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s),$$

where we consider F a function of all s such that the improper integral converges.
In the first problem you will show that this transform is *linear*, that is, that for constants a, b and functions $f(t)$ and $g(t)$ we have

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[f(t)].$$

To aid you in this endeavor, we demonstrate how to show $\mathcal{L}[af(t)] = a\mathcal{L}[f(t)]$. To see this, notice

$$\mathcal{L}[af(t)] = \int_0^\infty e^{-st}(af(t))\, dt = a\int_0^\infty e^{-st} f(t)\, dt = a\mathcal{L}[f(t)].$$

This verifies what is desired.

1. Prove the Laplace transform is linear. That is, show that

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[f(t)]$$

for arbitrary functions $f(t)$ and $g(t)$ and constants a and b.

Solution: To see this, notice

$$\begin{aligned}\mathcal{L}[af(t) + bg(t)] = \int_0^\infty e^{-st}(af(t) + bg(t))\, dt &= \int_0^\infty ae^{-st} f(t) + be^{-st} g(t)\, dt \\ &= \int_0^\infty ae^{-st} f(t) + be^{-st} g(t)\, dt \\ &= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].\end{aligned}$$

This verifies what is desired.

For the next problem, recall how we have shown that $\mathcal{L}[f'(t)] = sF(s) - f(0)$. Via integration by parts we have

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt.$$

where $u = e^{-st}$ and $dv = f'(t)\, dt$. We may assume, as usual, that $f(t)$ grows slower than the exponential so that

$$e^{-st} f(t)\Big|_0^\infty = \lim_{b\to\infty} e^{-sb} f(b) - e^0 f(0) = 0 - f(0) = -f(0).$$

Considering the other term we notice

$$\int_0^\infty (-se^{-st}) f(t)\, dt = -s\int_0^\infty e^{-st} f(t)\, dt = -s\mathcal{L}[f(t)].$$

Putting these two observations together yields

$$e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt = -f(0) - (-s\mathcal{L}[f(t)]) = s\mathcal{L}[f(t)] - f(0) = sF(s) - f(0).$$

Thus, by connecting the appropriate string of equalities, we have shown

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

2. Prove $\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$.

Hint: Follow steps similar to the above discussion. You may assume that f and all of it's derivatives grow slower than the exponential function e^{st}. You will need to use integration by parts twice *or* only once and then use a fact we have already shown.

Solution: To see this, notice that integration by parts yields

$$\mathcal{L}[f''(t)] = \int_0^\infty e^{-st} f''(t)\, dt = e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f'(t)\, dt.$$

where $u = e^{-st}$ and $dv = f''(t)\, dt$. As above we have

$$e^{-st} f'(t)\Big|_0^\infty = \lim_{b\to\infty} e^{-sb} f'(b) - e^0 f'(0) = 0 - f'(0) = -f'(0).$$

Considering the other term we see

$$\int_0^\infty (-se^{-st}) f'(t)\, dt = -s \int_0^\infty e^{-st} f'(t)\, dt = -s\mathcal{L}[f'(t)].$$

But from above we know that $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$. So

$$-s\mathcal{L}[f'(t)] = -s(sF(s) - f(0)) = -s^2F(s) + sf(0).$$

Putting this all together gives us

$$e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f'(t)\, dt = -f'(0) - (-s^2F(s) + sf(0)) = s^2F(s) - sf(0) - f'(0).$$

Connecting this last equality with the first verifies that

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0).$$

3. Show that $\mathcal{L}[e^{at} f(t)] = F(s - a)$.

Hint: If

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

then

$$F(15) = \int_0^\infty e^{-(15)t} f(t)\, dt.$$

Indeed, for any input k, we have

$$F(k) = \int_0^\infty e^{-(k)t} f(t)\, dt.$$

With this in mind, what is $F(s - a)$? Knowing that should help you in your verification.

Solution: We first note that

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{(a-s)t} f(t)\, dt.$$

Next, we see

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}(e^{at}f(t))\,dt = \int_0^\infty e^{-st+at}f(t)\,dt = \int_0^\infty e^{(a-s)t}f(t)\,dt = F(s-a).$$

Thus, as desired, we have shown

$$\mathcal{L}[e^{at}f(t)] = F(s-a).$$

4. Find the general solution of this DE using the Laplace transform

$$y'' + 9y = e^t.$$

Hint: It may help to let c_1 and c_2 be arbitrary constants for which you set $y(0) = c_1$ and $y'(0) = c_2$.

Solution: To begin we take the transform of the equation and solve for $\mathcal{L}[y] = Y(s)$

$$\begin{aligned}
\mathcal{L}[y'' + 9y] = \mathcal{L}[e^t] &\Longrightarrow \mathcal{L}[y''] + 9\mathcal{L}[y] = \frac{1}{s-1} \\
&\Longrightarrow s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{1}{s-1} \\
&\Longrightarrow s^2Y(s) - sc_1 - c_2 + 9Y(s) = \frac{1}{s-1} \\
&\Longrightarrow s^2Y(s) - sc_1 - c_2 + 9Y(s) = \frac{1}{s-1} \\
&\Longrightarrow (s^2+9)Y(s) = \frac{1}{s-1} + sc_1 + c_2 \\
&\Longrightarrow Y(s) = \frac{1}{(s-1)(s^2+9)} + \frac{sc_1}{(s^2+9)} + \frac{c_2}{(s^2+9)}.
\end{aligned}$$

Via partial fraction decomposition we have that

$$Y(s) = \left(\frac{1}{10}\right)\frac{1}{s-1} + \left(c_1 - \frac{1}{10}\right)\frac{s}{(s^2+9)} + \left(c_2 - \frac{1}{10}\right)\frac{1}{(s^2+9)}.$$

To determine $y(t)$ we take the inverse Laplace transform of $Y(s)$:

$$\begin{aligned}
\mathcal{L}^{-1}[Y(s)] &= \mathcal{L}^{-1}\left[\left(\frac{1}{10}\right)\frac{1}{s-1} + \left(c_1 - \frac{1}{10}\right)\frac{s}{(s^2+9)} + \left(c_2 - \frac{1}{10}\right)\frac{1}{(s^2+9)}\right] \\
&= \left(\frac{1}{10}\right)\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \left(c_1 - \frac{1}{10}\right)\mathcal{L}^{-1}\left[\frac{s}{(s^2+9)}\right] + \left(c_2 - \frac{1}{10}\right)\mathcal{L}^{-1}\left[\frac{1}{(s^2+9)}\right] \\
&= \left(\frac{1}{10}\right)\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \left(c_1 - \frac{1}{10}\right)\mathcal{L}^{-1}\left[\frac{s}{(s^2+9)}\right] + \left(c_2 - \frac{1}{10}\right)\left(\frac{1}{3}\right)\mathcal{L}^{-1}\left[\frac{3}{(s^2+9)}\right]
\end{aligned}$$

$$\Longrightarrow \quad y(t) = \left(\frac{1}{10}\right)e^t + \left(c_1 - \frac{1}{10}\right)\cos 3t + \left(c_2 - \frac{1}{10}\right)\left(\frac{1}{3}\right)\sin 3t$$

While this is a perfectly acceptable form of the general solution, we note that as c_1 and c_2 are arbitrary constants we can freely set

$$k_1 = c_1 - \frac{1}{10} \text{ and } k_2 = \left(c_2 - \frac{1}{10}\right)\left(\frac{1}{3}\right).$$

In conclusion, the general solution of this differential equation is

$$y(t) = \frac{e^t}{10} + k_1 \cos 3t + k_2 \sin 3t$$

for arbitrary constants k_1 and k_2.