Math 2410-004/013 **Name:** Solutions **Due:** 4/20/18

Homework #6: Chapter 4

Recall that an n-th order initial value problem is the to solve

$$a_n(x)y^{(n)} + \cdots + a_1(x)y'(x) + a_0(x)y = g(x)$$

subject to the n conditions

$$y(x_0) = y_0, \quad y'(x_1) = y_1, \quad \ldots, \quad y^{(n)}(x_n) = y_n.$$

Here we discuss how to solve a second order initial value problem

$$a_2y'' + a_1y' + a_0y = g(x), \qquad y(x_0) = y_0, \quad y'(x_1) = y_1$$

when the DE in question is a linear differential equation. Just as in the first-order case, the method of solving such an initial value problem is to first obtain the general solution of the DE and then determine values of the parameters in that general solution which satisfy the initial conditions.
For example, to solve the initial value problem

$$y'' - 4y' = 12x, \qquad y(0) = 4, \quad y'(0) = 1$$

we begin by noting that $y(x) = c_1e^{2x} + c_2e^{-2x} - 3x$ is the general solution. In order for $y(0) = 4$, we must have c_1 and c_2 such that

$$y(0) = c_1 + c_2 = 4.$$

In order for $y'(0) = 1$, we must have c_1 and c_2 such that

$$y'(0) = 2c_1 - 2c_2 - 3 = 1.$$

This gives us the system of equations

$$\begin{aligned} c_1 + c_2 &= 4 \\ 2c_1 - 2c_2 &= 4 \end{aligned}$$

which is satisfied by $c_1 = 3$ and $c_2 = 1$. So $y(x) = 3e^{2x} + e^{-2x} - 3x$ is the solution of the given IVP.

1. Solve the IVP

$$y'' - 4y' - 5y = 0, \qquad y(1) = 2, \quad y'(1) = 2$$

Solution: This differential equation is homogeneous so to find the general solution we consider the roots of the auxiliary equation

$$m^2 - 4m - 5 = (m-5)(m+1) = 0.$$

Seeing as they are $m_1 = 5$ and $m_2 = -1$ we have

$$y(x) = c_1e^{5x} + c_2e^{-x}$$

as the general solution to this DE. To ensure y satisfies the initial conditions, we choose c_1 and c_2 such that

$$\begin{aligned} y(1) &= c_1e^5 + c_2e^{-1} = 2, \\ y'(1) &= 5c_1e^5 - c_2e^{-1} = 2. \end{aligned}$$

Adding these equations together yields that $4c_1e^5 = 4$ so that $c_1 = e^{-5}$. This implies that $e^{-5}e^5 + c_2 e{-}1 = 2$, that is, that $c_2e^{-1} = 1$. Hence $c_2 = e$.

Thus, the solution to this IVP is

$$y(x) = e^{-5} \cdot e^{5x} + e \cdot e^{-x} = e^{5(x-1)} + e^{1-x}.$$

2. Solve the IVP

$$y'' + 4y = -2, \qquad y(\pi/8) = \frac{1}{2}, \quad y'(\pi/8) = 2$$

Solution: This differential equation is nonhomogeneous so to find the general solution we begin by considering the associated homogeneous equation: $y'' + 4y = 0$. Considering the auxiliary equation,

$$m^2 + 4 = 0,$$

which has $m = \pm 2i$ for roots, we see that that the complimentary function is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x.$$

We next require a particular solution to $y'' + 4y = -2$. We could use undetermined coefficients here with the trial solution $y_p = A$ but it is rather easy to see a particular solution should $y_p = (-1/2)$. Hence, the general solution to this DE is

$$y(x) = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}.$$

To ensure y satisfies the initial conditions, we choose c_1 and c_2 such that

$$\begin{aligned} y\left(\frac{\pi}{8}\right) = c_1 \cos 2\left(\frac{\pi}{8}\right) + c_2 \sin 2\left(\frac{\pi}{8}\right) - \frac{1}{2} = c_1 \cos\left(\frac{\pi}{4}\right) + c_2 \sin\left(\frac{\pi}{4}\right) - \frac{1}{2} \\ = c_1\frac{\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} y'\left(\frac{\pi}{8}\right) = -2c_1 \sin 2\left(\frac{\pi}{8}\right) + 2c_2 \cos 2\left(\frac{\pi}{8}\right) = -2c_1 \sin\left(\frac{\pi}{4}\right) + 2c_2 \cos\left(\frac{\pi}{4}\right) \\ = -2c_1\frac{\sqrt{2}}{2} + 2c_2\frac{\sqrt{2}}{2} = 2 \end{aligned}$$

Simplifying these equations gives rise to the system

$$\begin{aligned} c_1 + c_2 &= \frac{2}{\sqrt{2}} = \sqrt{2} \\ -c_1 + c_2 &= \frac{2}{\sqrt{2}} = \sqrt{2} \end{aligned}$$

Quickly, we can find that $c_1 = 0$ and $c_2 = \sqrt{2}$.

Thus, the solution to this IVP is

$$y(x) = \sqrt{2} \sin 2x - \frac{1}{2}.$$

A similar type of problem that we can discuss for higher order differential equations is a *boundary value problem*. Here, as opposed to giving initial conditions for a function and each of its derivatives, one is asked to find a function which satisfies certain values on the boundary of its domain. Specifically, a second-order boundary value problem (BVP) is the task of finding a function $y(x)$ defined on the interval $[x_0, x_1]$ such that

$$a_2 y'' + a_1 y' + a_0 y = g(x), \qquad y(x_0) = y_1, \quad y(x_1) = y_1.$$

Here we are given values for y to satisfy on the endpoints of its interval of definition.
As an example, we solve the boundary value problem

$$y'' + 9y = 9x, \qquad y(0) = 12, \quad y(\pi/6) = 7\pi/6.$$

We begin by noting that the general solution of the DE in question is $y(x) = c_1 \cos(3x) + c_2 \sin(3x) + x$. In order for $y(0) = 12$, we must have c_1 and c_2 such that

$$y(0) = c_1 = 12.$$

In order for $y(\pi/6) = 7\pi/6$, we must have c_1 and c_2 such that

$$y'(\pi/6) = c_2 + \frac{\pi}{6} = 7\pi/6.$$

This gives us the system of equations

$$\begin{aligned} c_1 &= 12 \\ c_2 + \frac{\pi}{6} &= 7\pi/6 \end{aligned}$$

which is satisfied by $c_1 = 12$ and $c_1 = 1$. So $y(x) = 12\cos(3x) + \sin(3x) + x$ is the solution of the given BVP.

3. Solve the following BVP:

$$y'' - 10y' + 25y = 0, \qquad y(0) = 1, \quad y(1) = 0.$$

Solution: This differential equation is homogeneous so to find the general solution we consider the roots of the auxiliary equation

$$m^2 - 10m + 25 = (m-5)^2 = 0.$$

Seeing as there is only the root $m = 5$ of multiplicity 2, we have

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}$$

as the general solution to this DE. To ensure y satisfies the initial conditions, we choose c_1 and c_2 such that

$$\begin{aligned} y(0) &= c_1 = 1, \\ y(1) &= c_1 e^5 + c_2 e^5 = 0. \end{aligned}$$

Since $c_1 = 1$, we see $c_2 = -1$.
Thus, the solution to this BVP is

$$y(x) = e^{5x} - x e^{5x}.$$

4. Solve the following BVP:

$$y'' - y = x^2 + 1, \qquad y(0) = -2, \quad y(1) = e - 4.$$

Solution: This differential equation is nonhomogeneous so to find the general solution we begin by considering the associated homogeneous equation: $y'' - y = 0$. Considering the auxiliary equation,

$$m^2 - 1 = (m+1)(m-1) = 0,$$

which has $m_1 = -1$ and $m_2 = 1$ for roots, we see that that the complimentary function is

$$y_c(x) = c_1 e^{-x} + c_2 e^x.$$

We next require a particular solution to $y'' - y = x^2 + 1$. To find this, we use undetermined coefficients. As $g(x) = x^2 + 1$ in this case, we use for a trial solution a generic quadratic polynomial. Namely, $y_p = Ax^2 + Bx + C$. Thus $y_p'' = 2A$. And we have that y_p is a solution if A, B and C are such that

$$2A - (Ax^2 + Bx + C) = (-A)x^2 + (-B)x + (2A - C) = x^2 + 1.$$

So $-A = 1$, $B = 0$ and $2A - C = 1$. Clearly $A = -1$, $B = 0$ and $C = -3$

Hence, the general solution to this DE is

$$y(x) = y_c + y_p = c_1 e^{-x} + c_2 e^x - x^2 - 3$$

To ensure y satisfies the initial conditions, we choose c_1 and c_2 such that

$$\begin{aligned} y(0) &= c_1 + c_2 - 3 = -2 \\ y(1) &= c_1 e^{-1} + c_2 e - 4 = e - 4 \end{aligned}$$

Here if $c_1 = 0$ and $c_2 = 1$ the boundary conditions are satisfied.

Thus, the solution to this BVP is

$$y(x) = e^x - x^2 - 3$$

Now, unlike IVPs, BVPs do not necessarily have a solution (even when the DE in question satisfies some existence and uniqueness theorems). To see this, we present one more example: that there is no solution to the BVP

$$y'' + 9y = 9x, \qquad y(0) = 1, \quad y(\pi/3) = \pi/3.$$

From above, we already know that the general solution to this DE is $y(x) = c_1 \cos(3x) + c_2 \sin(3x) + x$. So, to satisfy the boundary conditions we require c_1 and c_2 such that

$$y(0) = c_1 = 1$$

and

$$y(\pi/3) = -c_1 + \frac{\pi}{3} = \frac{\pi}{3}.$$

That is, c_1 and c_2 that satisfy

$$\begin{aligned} c_1 &= 1 \\ -c_1 &= 0. \end{aligned}$$

This would require c_1 to both be 1 and 0, and as this is not possible, we conclude there *are no such c_1 and c_2 that make the general solution solve the BVP*. Hence, it has no solution.

5. Show that the BVP

$$y'' - 2y' + 2y = 0, \qquad y(0) = 1, \quad y(\pi) = 1$$

has no solution.

Solution: We begin as usual by finding the general solution. As this DE is homogeneous, we consider the auxiliary equation

$$m^2 - 2m + 2 = 0$$

which has roots $m = 1 \pm i$. Hence, the general solution is

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x.$$

To satisfy the boundary conditions we need c_1 and c_2 such that

$$\begin{aligned} y(0) &= c_1 = 1 \\ y(\pi) &= -c_1 e^{\pi} = 1 \end{aligned}$$

But there is clearly no such c_1 and c_2 as this would require $c_1 = 1$ and $c_1 = -e^{-\pi}$ but $1 \neq -e^{-\pi}$. Hence, there can be no solution to this BVP.