Math 2410-004/013 **Name:** ____________________ **Due:** 4/27/18

Homework #7: The Laplace transform

In this homework you will verify a few properties of the Laplace transform. Recall the definition is as follows

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s),$$

where we consider F a function of all s such that the improper integral converges.
In the first problem you will show that this transform is *linear*, that is, that for constants a, b and functions $f(t)$ and $g(t)$ we have

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[f(t)].$$

To aid you in this endeavor, we demonstrate how to show $\mathcal{L}[af(t)] = a\mathcal{L}[f(t)]$. To see this, notice

$$\mathcal{L}[af(t)] = \int_0^\infty e^{-st}(af(t))\, dt = a\int_0^\infty e^{-st} f(t)\, dt = a\mathcal{L}[f(t)].$$

This verifies what is desired.

1. Prove the Laplace transform is linear. That is, show that

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[f(t)]$$

for arbitrary functions $f(t)$ and $g(t)$ and constants a and b.

For the next problem, recall how we have shown that $\mathcal{L}[f'(t)] = sF(s) - f(0)$. Via integration by parts we have

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt.$$

where $u = e^{-st}$ and $dv = f'(t)\, dt$. We may assume, as usual, that $f(t)$ grows slower than the exponential so that

$$e^{-st} f(t)\Big|_0^\infty = \lim_{b\to\infty} e^{-sb} f(b) - e^0 f(0) = 0 - f(0) = -f(0).$$

Considering the other term we notice

$$\int_0^\infty (-se^{-st}) f(t)\, dt = -s\int_0^\infty e^{-st} f(t)\, dt = -s\mathcal{L}[f(t)].$$

Putting these two observations together yields

$$e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt = -f(0) - (-s\mathcal{L}[f(t)]) = s\mathcal{L}[f(t)] - f(0) = sF(s) - f(0).$$

Thus, by connecting the appropriate string of equalities, we have shown

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

2. Prove $\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$.

   Hint: Follow steps similar to the above discussion. You may assume that f and all of it's derivatives grow slower than the exponential function e^{st}. You will need to use integration by parts twice *or* only once and then use a fact we have already shown.

3. Show that $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

   Hint: If

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

   then

$$F(15) = \int_0^\infty e^{-(15)t} f(t)\, dt.$$

   Indeed, for any input k, we have

$$F(k) = \int_0^\infty e^{-(k)t} f(t)\, dt.$$

   With this in mind, what is $F(s-a)$? Knowing that should help you in your verification.

4. Find the general solution of this DE using the Laplace transform

$$y'' + 9y = e^t.$$

   Hint: It may help to let c_1 and c_2 be arbitrary constants for which you set $y(0) = c_1$ and $y'(0) = c_2$.