# An eleventh order homogeneous linear equation.

Find the general solution of the following differential equation

$$y^{(11)}+9y^{(10)}+13y^{(9)}-85y^{(8)}-339y^{(7)}-585y^{(6)}+107y^{(5)}+4413y^{(4)}+10106y'''+14680y''+480y'-28800y=0.$$

You may use the fact that the auxiliary equation

$$m^{11}+9m^{10}+13m^9-85m^8-339m^7-585m^6+107m^5+4413m^4+10106m^3+14680m^2+480m-28800=0$$

factors into

$$(m-1)(m+2)(m^2+5)(m^2+2m+5)(m-3)^2(m+4)^3=0.$$

**Solution:** Recall that the general solution of an $n$-th order linear homogeneous equation with constant coefficients can be determined directly from the auxiliary equation. Specifically, each real root $m_i$ of the auxiliary equation gives a linearly independent solution

$$y_i(x)=e^{m_i x}.$$

Each complex pair of roots $m=a\pm bi$ gives two linearly independent solutions

$$y_j(x)=e^{ax}\cos bx \text{ and } y_{j+1}(x)=e^{ax}\sin(ax).$$

And finally, for a repeated root $m_\ell$ of multiplicity $k$, we obtain $k$ linearly independent solutions

$$y_\ell(x)=e^{m_\ell x},\quad y_{\ell+1}=xe^{m_\ell x},\quad y_{\ell+2}=x^2e^{m_\ell x},\quad \ldots,\quad y_{\ell+(k-2)}=x^{k-2}e^{m_\ell x},\quad y_{\ell+(k-1)}=x^{k-1}e^{m_\ell x}.$$

Together, these will account for the $n$ linearly independent solutions needed to form a fundamental set of solutions.
Here, the auxiliary equation of the DE in question is

$$(m-1)(m+2)(m^2+5)(m^2+2m+5)(m-3)^2(m+4)^3=0.$$

In this case, we see there are two real roots (red), two complex pairs of roots (blue) and two repeated roots (green) based upon the factors in the auxiliary equation.
From the two factors $(m-1)$ and $(m+2)$ we obtain real roots $m_1=1$ and $m_2=-2$ respectively. These determine the two linearly independents solutions

$$y_1(x)=e^x \text{ and } y_2(x)=e^{-2x}.$$

The factor $(m^2+5)$ yields the complex pair of roots $\pm i\sqrt{5}$ from which we obtain two linearly independent solutions

$$y_3(x)=\cos\sqrt{5}x \text{ and } y_4=\sin\sqrt{5}x.$$

The factor $(m^2+2m+5)$ yields the complex pair of roots $-1\pm 2i$ from which we obtain two linearly independent solutions

$$y_5(x)=e^{-x}\cos 2x \text{ and } y_6=e^{-x}\sin 2x.$$

The factor $(m-3)^2$ determines a repeated root $m_7 = 3$ of multiplicity 2. Thus two linearly independent solutions to the DE are

$$y_7(x) = e^{3x} \text{ and } y_8(x) = xe^{3x}.$$

The factor $(m+4)^3$ determines a repeated root $m_9 = -4$ of multiplicity 3. Thus three linearly independent solutions to the DE are

$$y_9(x) = e^{-4x}, \quad y_{10}(x) = xe^{-4x} \text{ and } y_{11}(x) = x^2e^{-4x}.$$

The differential equation is of order 11 and we have 11 linearly independent solutions. Thus

$$y_1(x) = e^x, \quad y_2(x) = e^{-2x}, \quad y_3(x) = \cos\sqrt{5}x, \quad y_4(x) = \sin\sqrt{5}x, \quad y_5(x) = e^{-x}\cos 2x, \quad y_6 = e^{-x}\sin(2x),$$
$$y_7(x) = e^{3x}, \quad y_8(x) = xe^{3x}, \quad y_9(x) = e^{-4x}, \quad y_{10}(x) = xe^{-4x} \text{ and } y_{11}(x) = x^2e^{-4x}.$$

form a fundamental set of solutions to the DE. The general solution is then

$$\begin{aligned} y(x) &= c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + c_5y_5 + c_6y_6 + c_7y_7 + c_8y_8 + c_9y_9 + c_{10}y_{10} + c_{11}y_{11} \\ &= c_1e^x + c_2e^{-2x} + c_3\cos\sqrt{5}x + c_4\sin\sqrt{5}x + c_5e^{-x}\cos 2x + c_6e^{-x}\sin(2x) \\ &\qquad + c_7e^{3x} + c_8xe^{3x} + c_9e^{-4x} + c_{10}xe^{-4x} + c_{11}x^2e^{-4x} \end{aligned}$$

for arbitrary constants $c_1, c_2, \ldots, c_{10}, c_{11}$.