Math 2410Q-003/014

Exam 2

April 2^{nd}, 2018

Name: ______________________________

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: ______________________________
(full name)

Questions:	1	2	3	4	5	6	Bonus	**Total**
Score:								

Percentage

1. (8 points) Determine if $\mathbf{X}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$ is a solution to the system of first-order differential equations

$$\mathbf{X}'(t) = \begin{pmatrix} 4 & -1 \\ 7 & -4 \end{pmatrix} \mathbf{X}(t).$$

Be sure to justify your answer.

2. (8 points) Find the general solution to the first-order, linear system

$$\mathbf{X}'(t) = \begin{pmatrix} 5 & \alpha \\ 0 & \beta \end{pmatrix} \mathbf{X}(t)$$

if

(a) (2 points) $\alpha = 0,\ \beta = 1.$

(b) (2 points) $\alpha = 3,\ \beta = -2.$

(c) (4 points) $\alpha = 2,\ \beta = 5.$

3. (8 points) Consider the first-order system of nonhomogenous linear differential equations

$$\begin{aligned}\frac{dx}{dt} &= x - y + e^t \\ \frac{dy}{dt} &= -x + y - 2e^t\end{aligned}$$

(a) (2 points) Rewrite this system as a matrix equation.

(b) (6 points) The general solution to this system is

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(You **do not** need to verify this.)
Use this to solve the following initial value problem.

$$\begin{aligned}\frac{dx}{dt} &= x - y + e^t, \qquad x(0) = -1 \\ \frac{dy}{dt} &= -x + y - 2e^t, \quad y(0) = 2\end{aligned}$$

Hint: If $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ then $\mathbf{X}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$.

4. (8 points) Find the general solution of the following system of first-order linear differential equations

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix} \mathbf{X}.$$

You may use the fact that this coefficient matrix has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ with respective eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}.$$

Note: The coefficient matrix may have more eigenvalues than these two.

5. (8 points) Recall that two vectors are linearly dependent if and only if one is a constant multiple of the other.

   Suppose for three non-zero 2×1 column vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, we have

   $$W(\mathbf{u}, \mathbf{v}) = 0 \text{ and } W(\mathbf{v}, \mathbf{w}) = 0.$$

   Here W denotes the Wronskian.

   (a) (4 points) What can you say about the value of $W(\mathbf{u}, \mathbf{w})$? Justify your answer.

   (b) (4 points) Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are each solutions to a two dimensional system of first-order homogeneous linear differential equations

   $$\mathbf{X}' = A\mathbf{X}.$$

   Which of the following form a fundamental set of solutions to this system? Choose one and justify your choice.

   (i) The collection $\mathbf{u}$ and $\mathbf{v}$.
   (ii) The collection $\mathbf{u}$ and $\mathbf{w}$.
   (iii) The collection $\mathbf{v}$ and $\mathbf{w}$.
   (iv) The collection $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
   (v) None of the above.

6. (8 points) Find the general solution of the following system of first-order linear differential equations

$$\frac{dx}{dt} = 6x - y$$
$$\frac{dy}{dt} = 5x + 2y$$

7. (4 points (bonus)) Verify the superposition principle for two-dimensional first-order systems of homogeneous linear differential equations.

   That is, suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions to the linear system

   $$\mathbf{X}' = A\mathbf{X}$$

   and show that for any constants c_1 and c_2 the function $\mathbf{X}_3 = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$ is also a solution to the system.

   You may use any facts about matrix arithmetic without justification.

8. (4 points (bonus)) Suppose that a matrix A with real entries has the complex eigenvalues $\lambda = a \pm bi$ with $b \neq 0$. Suppose also that

   $$\mathbf{V}_0 = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}$$

   is an eigenvector with eigenvalue $\lambda_0 = a + bi$. Show that

   $$\mathbf{V}_1 = \begin{pmatrix} x_1 - iy_1 \\ x_2 - iy_2 \end{pmatrix}$$

   is an eigenvector with eigenvalue $\lambda_1 = a - bi$.

   In other words, show that the complex conjugate of an eigenvector with eigenvalue λ, is an eigenvector with eigenvalue $\overline{\lambda}$, the complex conjugate of λ.

   Hint: Two complex numbers $c_0 + d_0 i$ and $c_1 + d_1 i$ are equal if and only if $c_0 = c_1$ and $d_0 = d_1$. That is, if their real and imaginary parts are equal.