Math 2410Q-003/014 **Exam 2** **April 2^{nd}, 2018**

Name: Solutions

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: ______________________________
(full name)

Questions:	1	2	3	4	5	6	Bonus	**Total**
Score:								

Percentage

1. (8 points) Determine if $\mathbf{X}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$ is a solution to the system of first-order differential equations

$$\mathbf{X}'(t) = \begin{pmatrix} 4 & -1 \\ 7 & -4 \end{pmatrix} \mathbf{X}(t).$$

Be sure to justify your answer.

Solution: We first check that $\mathbf{X}_1$ satisfies the system in question. To begin, the left-hand side of the differential equation (with $\mathbf{X}_1$ substituted in) is

$$\mathbf{X}_1' = \frac{d}{dt} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} = -3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} e^{-3t}.$$

The right-hand side of the differential equation (with $\mathbf{X}_1$ substituted in) is

$$\begin{pmatrix} 4 & -1 \\ 7 & -4 \end{pmatrix} \mathbf{X}_1(t) = \begin{pmatrix} 4 & -1 \\ 7 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} = \begin{pmatrix} 4-1 \\ 7-4 \end{pmatrix} e^{-3t} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{-3t}.$$

Since

$$\begin{pmatrix} -3 \\ -3 \end{pmatrix} e^{-3t} \neq \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^{-3t},$$

we see that the left-hand side and the right-hand side of the system of differential equations is not equal upon substituting $\mathbf{X}_1$ in. Thus, $\mathbf{X}_1$ *is not* a solution to the system.

2. (8 points) Find the general solution to the first-order, linear system

$$\mathbf{X}'(t) = \begin{pmatrix} 5 & \alpha \\ 0 & \beta \end{pmatrix} \mathbf{X}(t)$$

if

(a) (2 points) $\alpha = 0,\ \beta = 1$.

(b) (2 points) $\alpha = 3,\ \beta = -2$.

(c) (4 points) $\alpha = 2,\ \beta = 5$.

Solution: To find the general solution of any linear system of differential equations, we first find the eigenvalues and eigenvectors of the coefficient matrix. In this case, the coefficient matrix of the system varies for each part of this question. Before specifying to any part of this question, we find the eigenvalues and eigenvectors of the coefficient matrix in general. To that end, we see that the characteristic polynomial of the coefficient matrix is

$$\det \begin{pmatrix} 5-\lambda & \alpha \\ 0 & \beta-\lambda \end{pmatrix} = (5-\lambda)(\beta-\lambda).$$

Thus, the characteristic equation of this matrix is $(5-\lambda)(\beta-\lambda) = 0$ and we see the eigenvalues of this matrix are $\lambda_1 = 5$ and $\lambda_2 = \beta$.

The eigenvectors for λ_1 and λ_2 then are, respectively, nontrivial solutions to the systems associated with the augmented matrices

$$\left(\begin{array}{cc|c} 5-5 & \alpha & 0 \\ 0 & \beta-5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 0 & \alpha & 0 \\ 0 & \beta-5 & 0 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} 5-\beta & \alpha & 0 \\ 0 & \beta-\beta & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 5-\beta & \alpha & 0 \\ 0 & 0 & 0 \end{array}\right).$$

The first tells us that eigenvectors with eigenvalue 5 are nonzero vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $\alpha y = 0$ and $(\beta - 5)y = 0$. The second tells us that eigenvectors with eigenvalue β are nonzero vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $(5-\beta)x + \alpha y = 0$.

With this calculation made, we turn our attention to each part:

(a) Here $\alpha = 0$ and $\beta = 1$. So the eigenvalues of the coefficient matrix are real and distinct: namely $\lambda_1 = 5$ and $\lambda_2 = 1$. By the above calculation we know and eigenvector for λ_1 is any $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $-4y = 0$. In particular, we may use $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For λ_2, we need a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $4x = 0$. Here $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ works.

Thus

$$\mathbf{X}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{X}_2(t) = e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are a pair of linearly independent solutions to the system and thereby form a fundamental set. Hence, the general solution in this case is

$$\mathbf{X}(t) = c_1 X_1(t) + c_2 \mathbf{X}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for arbitrary constants c_1 and c_2.

(b) Here $\alpha = 3$ and $\beta = -2$. So the eigenvalues of the coefficient matrix again are real and distinct. Here we have $\lambda_1 = 5$ and $\lambda_2 = -2$. By the above calculation we know and eigenvector for λ_1 is any $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $3y = 0$ and $-7y = 0$. Once again, we may use $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For λ_2, we need a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $7x + 3y = 0$. Here $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ works.

Thus

$$\mathbf{X}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{X}_2(t) = e^{-2t} \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

are a pair of linearly independent solutions to the system and thereby form a fundamental set. So the general solution in this case is

$$\mathbf{X}(t) = c_1 X_1(t) + c_2 \mathbf{X}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

for arbitrary constants c_1 and c_2.

(c) Here $\alpha = 2$ and $\beta = 5$ so the only eigenvalue to the coefficient matrix is $\lambda = 5$, which is a repeated eigenvalue of multiplicity 2. The eigenvectors of this eigenvalue are nonzero $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $2y = 0$ (by the above calculation). In this case, by observation, we see any two eigenvectors are multiples of each other and hence there is only one linearly independent eigenvector. Again, this is the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So

$$\mathbf{X}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a solution to the system. Another linearly independent solution is given by

$$\mathbf{X}_2(t) = te^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{5t} \mathbf{W}$$

where $\mathbf{W} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is any vector such that

$$\begin{pmatrix} 5-5 & 2 \\ 0 & 5-5 \end{pmatrix} \mathbf{W} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In particular, $\mathbf{W}$ with $w_1 = 0$ and $w_2 = (1/2)$ is such a vector. Hence,

$$\mathbf{X}_2(t) = te^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{5t} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

is a second linearly independent solution to the system. Thus $\mathbf{X}_1$ and $\mathbf{X}_2$ are a fundamental set of solutions. The general solution is then

$$\mathbf{X}(t) = c_1 \mathbf{X}_1(t) + c_2 \mathbf{X}_2(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \left[te^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{5t} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right]$$

for arbitrary constants c_1 and c_2.

3. (8 points) Consider the first-order system of nonhomogenous linear differential equations

$$\frac{dx}{dt} = x - y + e^t$$
$$\frac{dy}{dt} = -x + y - 2e^t$$

(a) (2 points) Rewrite this system as a matrix equation.

Solution: Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then

$$\mathbf{X}'(t) = A\mathbf{X} + e^t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

is the desired matrix equation.

(b) (6 points) The general solution to this system is

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(You **do not** need to verify this.)
Use this to solve the following initial value problem.

$$\frac{dx}{dt} = x - y + e^t, \qquad x(0) = -1$$
$$\frac{dy}{dt} = -x + y - 2e^t, \quad y(0) = 2$$

Hint: If $\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ then $\mathbf{X}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$.

Solution: Since the general solution of the system is given, we simply need find c_1 and c_2 such that

$$\mathbf{X}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

That is, c_1 and c_2 such that

$$c_1 + c_2 - 2 = -1$$
$$c_1 - c_2 + 1 = 2.$$

Clearly, we need $c_1 = 1$ and $c_2 = 0$. Thus

$$\mathbf{X}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

is the solution to the given IVP.

4. (8 points) Find the general solution of the following system of first-order linear differential equations

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix} \mathbf{X}.$$

You may use the fact that this coefficient matrix has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ with respective eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_2 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}.$$

Note: The coefficient matrix may have more eigenvalues than these two.

Solution: We begin by finding all the eigenvalues of the coefficient matrix. To that end, we require λ such that

$$\det \begin{pmatrix} 1-\lambda & 1 & 3 \\ 0 & 2-\lambda & 0 \\ 0 & 3 & 0-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(-\lambda) = 0$$

Clearly, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = 1$.

Since these are real and distinct, we may find three linearly independent solutions to the system

$$\mathbf{X}_1 = e^{\lambda_1}\mathbf{V}_1, \quad \mathbf{X}_2 = e^{\lambda_2}\mathbf{V}_2, \text{ and } \mathbf{X}_3 = e^{\lambda_3}\mathbf{V}_3$$

where V_i is an eigenvector λ_i. The general solution will then be all linear combinations of these solutions.

To that end, note we are given eigenvectors for $\lambda_1 = 0$ and $\lambda_2 = 2$. So

$$\mathbf{X}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{X}_2 = e^{2t}\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}.$$

To find $\mathbf{V}_3$, we require a *nontrivial* solution to the system of equations

$$\left(\begin{array}{ccc|c} 1-\lambda_3 & 1 & 3 & 0 \\ 0 & 2-\lambda_3 & 0 & 0 \\ 0 & 3 & 0-\lambda_3 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right) \xrightarrow[R_3+(-3)R_1]{R_1+(-1)R_2} \left(\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right).$$

This equates to finding x, y and z such that

$$3z = 0, \quad y = 0, \text{ and } -z = 0$$

while at least one of these variables is non-zero. We see $y = 0$ and $z = 0$, so any choice of $x \neq 0$ will suffice. To maintain simplicity, we choose $x = 1$ and conclude

$$\mathbf{V}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is an eigenvector with eigenvalue $\lambda_3 = 1$. Hence,

$$\mathbf{X}_3 = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is the third and final linearly independent solution we need to form the general solution of this system.

In conclusion,

$$\mathbf{X}(t) = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3 = c_1 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5. (8 points) Recall that two vectors are linearly dependent if and only if one is a constant multiple of the other.

Suppose for three non-zero 2×1 column vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, we have

$$W(\mathbf{u}, \mathbf{v}) = 0 \text{ and } W(\mathbf{v}, \mathbf{w}) = 0.$$

Here W denotes the Wronskian.

(a) (4 points) What can you say about the value of $W(\mathbf{u}, \mathbf{w})$? Justify your answer.

Solution: The fact that $W(\mathbf{u}, \mathbf{v}) = 0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. In other words, for some constant k, $\mathbf{u} = k\mathbf{v}$.

The fact that $W(\mathbf{v}, \mathbf{w}) = 0$ implies that $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent. In other words, for some constant ℓ, $\mathbf{v} = \ell\mathbf{w}$.

So, we have that $\mathbf{u} = k(\ell\mathbf{w})$, which implies that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. Hence, $W(\mathbf{u}, \mathbf{w}) = 0$.

(b) (4 points) Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are each solutions to a two dimensional system of first-order homogeneous linear differential equations

$$\mathbf{X}' = A\mathbf{X}.$$

Which of the following form a fundamental set of solutions to this system? Choose one and justify your choice.

(i) The collection $\mathbf{u}$ and $\mathbf{v}$.
(ii) The collection $\mathbf{u}$ and $\mathbf{w}$.
(iii) The collection $\mathbf{v}$ and $\mathbf{w}$.
(iv) The collection $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
(v) None of the above.

Solution: The correct answer here is (v). As any collection of these solutions is linearly dependent, none can form a fundamental set of solutions.

6. (8 points) Find the general solution of the following system of first-order linear differential equations

$$\frac{dx}{dt} = 6x - y$$
$$\frac{dy}{dt} = 5x + 2y$$

Solution: Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$. Then this system of equations can be rewritten $\mathbf{X}'(t) = A\mathbf{X}(t)$.

To find the general solution, we first find the eigenvalues of A:

$$\det(A - \lambda I) = \det \begin{pmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{pmatrix} = (6-\lambda)(2-\lambda) + 5 = \lambda^2 - 8\lambda + 17 = 0.$$

Thus, λ is an eigenvalue if

$$\lambda = \frac{8 \pm \sqrt{64 - 4(17)}}{2} = \frac{8 \pm \sqrt{64 - 68}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i.$$

As A has complex eigenvalues, we need only find an eigenvector for one of these, say $\lambda = 4 + i$. (As we will see in question 8, the other eigenvector will be the conjugate of whatever we find here.)

To find a complex eigenvector, we note, as usual that

$$\left(\begin{array}{cc|c} 6-\lambda & -1 & 0 \\ 5 & 2-\lambda & 0 \end{array}\right) = \left(\begin{array}{cc|c} 6-(4+i) & -1 & 0 \\ 5 & 2-(4+i) & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

(Here the last implication follows from the fact that, *that in the complex case*, these rows are a complex multiple of each other. Specifically, $R_1 = (2+i)R_2$.) So, any nontrivial solution, that is, nonzero x and y such that $(2-i)x = y$, of this system gives rise to a complex eigenvector with eigenvalue $\lambda = 4 + i$. For instance, letting $x = 1$ yields the vector

$$\mathbf{V} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}.$$

Thus, a complex solution of this system is

$$\mathbf{X}_c = e^{(2+i)t} \begin{pmatrix} 1 \\ 2-i \end{pmatrix}.$$

Using Euler's formula, we may rewrite this function as follows:

$$\begin{aligned} \mathbf{X}_c = e^{(2+i)t} \begin{pmatrix} 1 \\ 2-i \end{pmatrix} &= e^{2t}(\cos t + i \sin t) \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) \\ &= e^{2t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i \left[e^{2t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} \right]. \end{aligned}$$

Letting $\mathbf{X}_r = \mathrm{Re}(\mathbf{X}_c)$ and $\mathbf{X}_i = \mathrm{Im}(\mathbf{X}_c)$ yields two linearly independent *real* solutions to the system:

$$\mathbf{X}_r = e^{2t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} \text{ and } \mathbf{X}_i = e^{2t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

Thus, the general solution of this system is

$$\mathbf{X}(t) = c_1\mathbf{X}_r + c_2\mathbf{X}_i = c_1 e^{2t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

for arbitrary constants c_1 and c_2.

7. (4 points (bonus)) Verify the superposition principle for two-dimensional first-order systems of homogeneous linear differential equations.

   That is, suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions to the linear system

$$\mathbf{X}' = A\mathbf{X}$$

   and show that for any constants c_1 and c_2 the function $\mathbf{X}_3 = c_1\mathbf{X}_1 + c_2\mathbf{X}_2$ is also a solution to the system.

   You may use any facts about matrix arithmetic without justification.

Solution: To show this we need verify that $\mathbf{X}_3' = A\mathbf{X}_3$. To that end note that because $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions we have

$$\mathbf{X}_1' = A\mathbf{X}_1 \text{ and } \mathbf{X}_2' = A\mathbf{X}.$$

Because the derivative distributes across addition we have

$$\mathbf{X}_3' = (c_1\mathbf{X}_1 + c_2\mathbf{X}_2)' = (c_1\mathbf{X}_1)' + (c_2\mathbf{X}_2)'$$

and as constants factor out of the derivative we have

$$(c_1\mathbf{X}_1)' + (c_2\mathbf{X}_2)' = c_1(\mathbf{X}_1)' + c_2(\mathbf{X}_2)'.$$

As initially noted, since $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions

$$c_1(\mathbf{X}_1)' + c_2(\mathbf{X}_2)' = c_1(A\mathbf{X}_1) + c_2(A\mathbf{X}_2).$$

It is a basic fact of matrix arithmetic that we may move scalar multiples around any part of the product

$$c_1(A\mathbf{X}_1) + c_2(A\mathbf{X}_2) = A(c_1\mathbf{X}_1) + A(c_2\mathbf{X}_2).$$

From the previous homework, we know matrix multiplication distributes across addition so we may factor out A, and recognize and expression equal to $\mathbf{X}_3$:

$$A(c_1\mathbf{X}_1) + A(c_2\mathbf{X}_2) = A(c_1\mathbf{X}_1 + c_2\mathbf{X}_2) = A\mathbf{X}_3.$$

Connecting this string of equalities yields, as desired

$$\mathbf{X}_3' = A\mathbf{X}_3;$$

that is, that X_3 is a solution to the system.

8. (4 points (bonus)) Suppose that a matrix A with real entries has the complex eigenvalues $\lambda = a \pm bi$ with $b \neq 0$. Suppose also that

$$\mathbf{V}_0 = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}$$

is an eigenvector with eigenvalue $\lambda_0 = a + bi$. Show that

$$\mathbf{V}_1 = \begin{pmatrix} x_1 - iy_1 \\ x_2 - iy_2 \end{pmatrix}$$

is an eigenvector with eigenvalue $\lambda_1 = a - bi$.

In other words, show that the complex conjugate of an eigenvector with eigenvalue λ, is an eigenvector with eigenvalue $\overline{\lambda}$, the complex conjugate of λ.

Hint: Two complex numbers $c_0 + d_0 i$ and $c_1 + d_1 i$ are equal if and only if $c_0 = c_1$ and $d_0 = d_1$. That is, if their real and imaginary parts are equal.

Solution: Let $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. So

$$\mathbf{V}_0 = \mathbf{X} + i\mathbf{Y} \text{ and } \mathbf{V}_1 = \mathbf{X} - i\mathbf{Y}.$$

By assumption, we know

$$A\mathbf{V}_0 = \lambda V_0.$$

As

$$A\mathbf{V}_0 = A(\mathbf{X} + i\mathbf{Y}) = A\mathbf{X} + iA\mathbf{Y}$$

and

$$\lambda V_0 == (a + bi)(\mathbf{X} + i\mathbf{Y}) = (a\mathbf{X} - b\mathbf{Y}) + i(b\mathbf{X} + a\mathbf{Y})$$

we have

$$A\mathbf{X} + iA\mathbf{Y} = (a\mathbf{X} - b\mathbf{Y}) + i(b\mathbf{X} + a\mathbf{Y}).$$

So

$$A\mathbf{X} = a\mathbf{X} - b\mathbf{Y} \text{ and } A\mathbf{Y} = b\mathbf{X}_0 + a\mathbf{Y}.$$

Hence,

$$\begin{aligned} A\mathbf{V}_1 = A(\mathbf{X} - i\mathbf{Y}) = A\mathbf{X} - iA\mathbf{Y} &= (a\mathbf{X} - b\mathbf{Y}) - i(b\mathbf{X} + a\mathbf{Y}) \\ &= (a - bi)\mathbf{X} + (-b - ia)\mathbf{Y} \\ &= (a - bi)\mathbf{X} - i(a - bi)\mathbf{Y} \\ &= (a - bi)(\mathbf{X} - i\mathbf{Y}) \\ &= (a - bi)\mathbf{V}_1 = \overline{\lambda}\mathbf{V}_1. \end{aligned}$$

Thus we see that $\mathbf{V}_1$ is an eigenvector of A with the eigenvalue $\overline{\lambda}$.