Math 2410Q-003/014 | # Exam I | **Feb 19^{th}, 2018**

Name: Solutions

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: ______________________________
(full name)

Questions:	1	2	3	4	5	6	Bonus	**Total**
Score:								

Percentage

1. (8 points) Assume k is a real valued constant. For which value(s) of k is $y = x^k$ a solution to the differential equation

$$x^2y'' + 4xy' - 4y = 0.$$

If no such value exists, explain why not.

Solution: If $y = x^k$, then $y' = kx^{k-1}$ and $y'' = k(k-1)x^{k-2}$. Thus, we see the left hand side of the differential equation is

$$\begin{aligned} x^2y'' + 4xy' - 4y &= x^2(k(k-1)x^{k-2}) + 4x(kx^{k-1}) - 4(x^k) \\ &= k(k-1)x^k + 4kx^k - 4x^k \\ &= x^k(k(k-1) + 4k - 4) \\ &= x^k(k^2 + 3k - 4). \end{aligned}$$

Hence, for this expression to equal the right hand side, namely 0, we need k such that

$$k^2 + 3k - 4 = (k+4)(k-1) = 0.$$

Note that x^k cannot make the expression equal 0 as x can take on any value here. Clearly, the two permissible values of k such that $y = x^k$ is a solution are $k = -4$ and $k = 1$.

2. (8 points) Consider the autonomous first order differential equation

$$\frac{dy}{dx} = y - y^3.$$

(a) (4 points) Find the critical points of this differential equation and sketch its phase portrait.

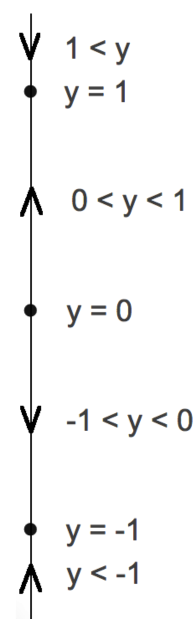
Solution:
Recall that critical points of an autonomous DE $dy/dx = f(y)$ are the values k such that $f(k) = 0$. Here

$$f(y) = y - y^3 = y(1-y^2) = y(1-y)(1+y) = 0$$

if $y = 0, \pm 1$. Hence $0, 1$ and -1 are the critical points of this DE. To sketch the phase portrait we look at the sign of $f(y)$ near these critical points:

$1 < y$	$f(y) < 0$
$0 < y < 1$	$f(y) > 0$
$-1 < y < 0$	$f(y) < 0$
$y < -1$	$f(y) > 0$

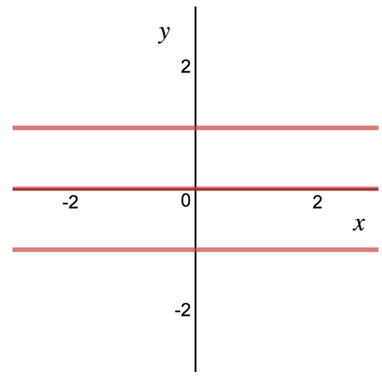
From this we generate the phase portrait given on the right.

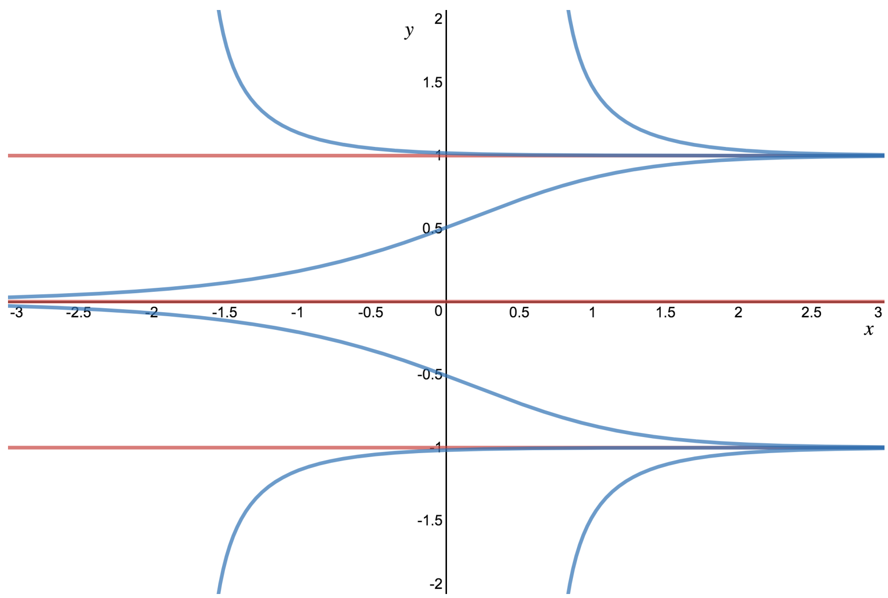


(b) (4 points) Sketch typical solution curves in the regions of the xy-plane determined by the graphs of the equilibrium solutions for this differential equation. (Make sure to include graphs of the equilibria themselves!)

Solution:

Based off our work in the previous part, we know that this differential equation has three equilibrium solutions, $y = 0$, $y = 1$ and $y = -1$. We plot these in red to the right. Regarding the phase portrait we see that solutions above $y = 1$ must tend towards 1; that solutions between 0 and 1 must tend to 1; that solutions between -1 and 0 tend to -1; and that solutions below -1 tend to -1. These observations allow us to generate a rough sketch of the behavior of solution curves in these regions of the plane. The desired plot is given below.





3. (8 points) Find the general solution to the linear, first order differential equation

$$\frac{dy}{dx} + \tan(x)y = \sec(x)$$

on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Your solution should be presented as an explicit function $y(x)$.

Hint. $\int \tan(x)dx = \ln(\sec(x)) + c.$

Solution: This is a linear differential equation in standard form with

$$P(x) = \tan(x) \text{ and } f(x) = \sec(x).$$

Both are continuous on the interval in question, $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus, via the integrating factor

$$\mu(x) = e^{\int \tan(x)\, dx} = e^{\ln(\sec(x))} = \sec(x)$$

we see that the general solution on I is given by

$$\begin{aligned} y(x) &= \frac{1}{\sec(x)} \int \sec(x)\sec(x)\, dx + \frac{c}{\sec(x)} \\ &= \cos(x) \int \sec^2(x)\, dx + c\cos(x) \\ &= \cos(x)\tan(x) + c\cos(x) \\ &= \sin(x) + c\cos(x). \end{aligned}$$

In summary, the general solution to this DE on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

$$y(x) = \sin(x) + c\cos(x).$$

4. (8 points) Determine a region of the xy-plane containing $(0,0)$ for which the differential equation

$$(4-y^2)y' = x^2$$

would have a unique solution whose graph passes through any point (x_0, y_0) in that region. Make sure to justify your answer.

Solution: To do this, we first find all regions for which the existence and uniqueness theorem applies and then determine the largest one in which $(0,0)$ lies. Recall a given IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

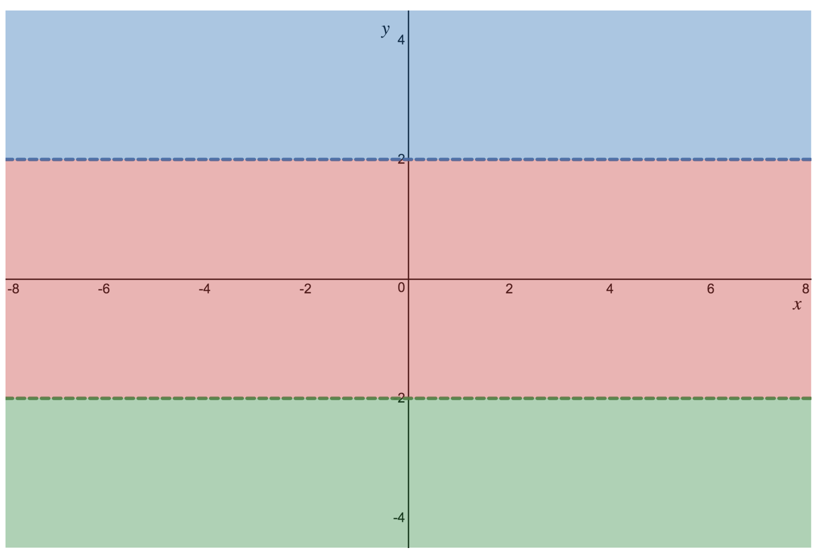
must have a unique solution if (x_0, y_0) is in a rectangle R in which f and $\partial f/\partial y$ are continuous. In this case, dividing by $(4-y^2)$ yields

$$\frac{dy}{dx} = \frac{x^2}{4-y^2}$$

from which we observe that

$$f(x,y) = \frac{x^2}{4-y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{2yx^2}{(4-y^2)^2}.$$

Clearly, both of these functions are continuous so long as $y \neq \pm 2$. Thus, existence and uniqueness applies at any point in the three regions $y > 2$, $y < -2$ and $-2 < y < 2$ (plotted below in blue, green and red respectively, note $y = \pm 2$ are dotted lines).



Only one of these includes $(0,0)$ so that is our answer: the region with $-\infty < x < \infty$ and $-2 < y < 2$.

5. (8 points) Consider the following initial value problem,

$$\frac{dy}{dx} = y(3 - xy), \quad y(0) = 1.$$

By hand, use Euler's method with $\Delta x = 1$ to approximate the value of $y(2)$. Use the table to record the necessary values of k, x_k, y_k, and $dy/dx = f(x_k, y_k)$ at each step. Justify your calculations if you wish to receive partial credit.

Solution:

k	x_k	y_k	$f(x_k, y_k) = y_k(3 - x_k y_k)$
0	0	1	$f(0,1) = 1(3 - 0 \cdot 1) = 3$
1	$0 + \Delta x = 1$	$1 + \Delta x f(0,1) = 1 + 1 \cdot 3 = 4$	$4(3 - 1 \cdot 4) = -4$
2	2	$4 + 1(-4) = 0$	

Thus, via our application of Euler's method we approximate the solution to this IVP to approximately output 0 on input 2. That is

$$y(2) \approx 0.$$

By the last bonus, we know the solution we are approximating is

$$y(x) = \frac{9e^{3x}}{e^{3x}(3x - 1) + 10}.$$

So we see $y(2) = 1.7911\ldots$. Hence our error is as follows

$$\begin{aligned}
\text{Absolute error} &= |1.7911 - 0| = 1.7911 \\
\text{Relative error} &= \left|\frac{1.7911 - 0}{1.7911}\right| = 1 \\
\text{Percent relative error} &= \left|\frac{1.7911 - 0}{1.7911}\right| \cdot 100 = 100\%
\end{aligned}$$

An error of 100%? Wow. This demonstrates how inaccurate such a large step size ($\Delta x = 1$) is.

6. (8 points) Consider the differential equation

$$\frac{dy}{dx} = \sin(x + y) - \sin(x - y).$$

(a) (2 points) Verify that this DE has the trivial solution $y(x) = 0$.

Solution: If $y = 0$, then $y' = 0$ and so

$$y' = 0 = \sin(x) - \sin(x) = \sin(x + 0) - \sin(x - 0) = \sin(x + y) - \sin(x - y).$$

Hence, the trivial solution $y = 0$ satisfies the DE and consequently is a solution on $(-\infty, \infty)$.

(b) (6 points) Use $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$ to solve this DE by separation of variables. Your solution may be presented as an implicit function.

Hint. $\displaystyle\int \csc(x)\,dx = -\ln(\cot(x) + \csc(x)) + c.$

Solution: Via the given identity, we can rewrite the right hand side as follows

$$\begin{aligned} \sin(x+y) - \sin(x-y) &= (\sin(x)\cos(y) + \cos(x)\sin(y)) - (\sin(x)\cos(y) - \cos(x)\sin(y)) \\ &= \sin(x)\cos(y) + \cos(x)\sin(y) - \sin(x)\cos(y) + \cos(x)\sin(y) \\ &= 2\cos(x)\sin(y) \end{aligned}$$

Thus, the given differential equation is separable and equivalent to

$$\frac{dy}{dx} = 2\cos(x)\sin(y).$$

Via separation of variables we have for any $y \neq k\pi$ with k an integer,

$$\int \csc(y)\,dy = \int 2\cos(x)\,dx \quad \Longrightarrow \quad -\ln(\cot(y) + \csc(y)) = 2\sin(x) + c.$$

Without the use of extensive knowledge of trigonometric identities or a computer algebra system, the best solution we can offer is an implicit one:

$$-\ln(\cot(y) + \csc(y)) = 2\sin(x) + c.$$

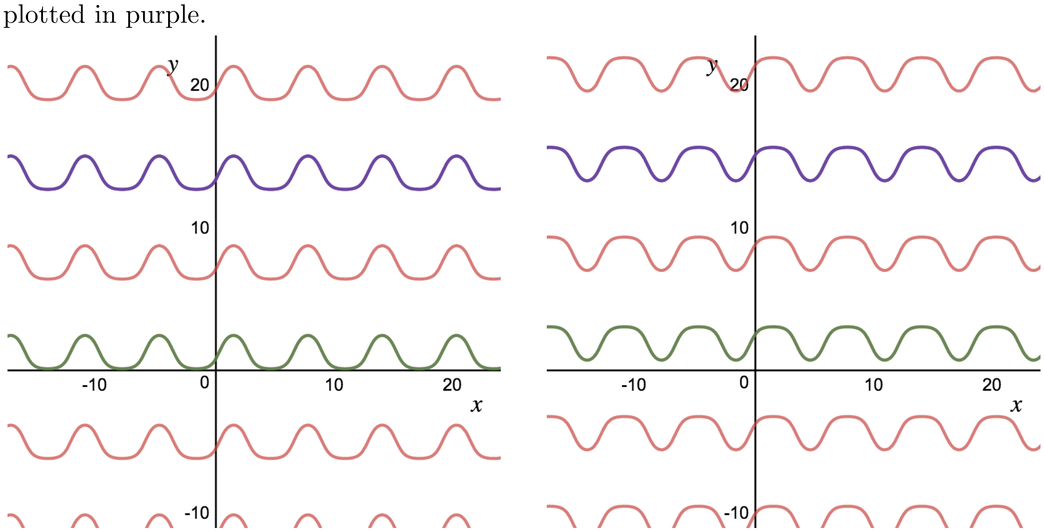
The following was not required but let's look at this family of implicit functions. What explicit functions are defined by this relation? Using WolframAlpha, we have that if

$$-\ln(\cot(y) + \csc(y)) = 2\sin(x) + c,$$

then

$$y(x) = 2\arctan\left(e^{2\sin(x)+c}\right) + 2n\pi$$

for some integer n. Plotted below you will find these functions with $-2 \leq n \leq 4$. On the left $c = -1$ and on the right $c = 1$. The specific function for $n = 0$ is plotted in green while $n = 2$ is plotted in purple.



7. (4 points) **Bonus:** Consider the following first order linear differential equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

Show that if $y_1(x)$ and $y_2(x)$ are solutions to the above equation, then the difference of these functions

$$y_3(x) = y_1(x) - y_2(x)$$

is a solution to associated linear equation

$$\frac{dy}{dx} + P(x)y = 0$$

in which $f(x) = 0$.

Solution: Note that because y_1 and y_2 are solutions to the first differential equation, we have

$$\frac{dy_1}{dx} + P(x)y_1 = f(x) \quad \text{and} \quad \frac{dy_2}{dx} + P(x)y_2 = f(x).$$

To verfiy that y_3 is a solution to the second differential equation, we need only show that

$$\frac{dy_3}{dx} + P(x)y_3 = 0.$$

Towards this end, note

$$\begin{aligned}
\frac{dy_3}{dx} + P(x)y_3 &= \frac{d(y_1 - y_2)}{dx} + P(x)(y_1 - y_2) \\
&= \frac{dy_1}{dx} - \frac{dy_2}{dx} + P(x)y_1 - P(x)y_2 \\
&= \left(\frac{dy_1}{dx} + P(x)y_1\right) - \left(\frac{dy_2}{dx} + P(x)y_2\right) \\
&= f(x) - f(x) \\
&= 0.
\end{aligned}$$

Thus, $y_3(x)$ is a solution to the DE

$$\frac{dy}{dx} + P(x)y = 0.$$

8. (4 points) **Bonus:** In question 5, you were asked to approximate the solution to the IVP

$$\frac{dy}{dx} = y(3 - xy), \quad y(0) = 1$$

at $x = 2$. Find the explicit solution to this IVP.

Solution: Rewriting this differential equation as

$$\frac{dy}{dx} - 3y = -xy^2$$

reveals it to be a Bernoulli equation. Hence, to solve this equation, we may use the substitution $u = y^{1-2} = y^{-1}$ to reduce this to a first order linear equation. Since $u = 1/y$, we have $y = 1/u$ and thus $dy/dx = (-1/u^2)(du/dx)$. Substituting this into the differential equation yields

$$\frac{-1}{u^2}\frac{du}{dx} - \frac{3}{u} = \frac{-x}{u^2}$$

To clear the denominator, we multiply the equation by $-u^2$ and obtain the linear equation

$$\frac{du}{dx} + 3u = x.$$

Using an integrating factor of $e^{\int 3\,dx} = e^{3x}$ yields the general solution

$$u(x) = e^{-3x}\int xe^{3x}\,dx + c_1 e^{-3x}$$

on $(-\infty, \infty)$. To simplify the expression on the right hand side, we use integration by parts with

$$\begin{aligned} u &= x & v &= \frac{e^{3x}}{3} \\ du &= x\,dx & dv &= e^{3x}\,dx \end{aligned}$$

to evaluate the integral:

$$\int xe^{3x}\,dx = \frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3} = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

Hence, we have

$$u(x) = \frac{x}{3} - \frac{1}{9} + c_1 e^{-3x} = \frac{e^{3x}(3x-1)+c}{9e^{3x}}.$$

Because $y = 1/u$, we obtain the general solution to this DE

$$y(x) = \frac{9e^{3x}}{e^{3x}(3x-1)+c}.$$

(It is worth noting that depending on c, the interval of definition for this solution will not be $(-\infty, \infty)$.) To finish solving the IVP, we need to choose c so that $y(x)$ satisfies the initial condition, $y(0) = 1$. Towards that end, note

$$y(0) = \frac{9e^0}{e^0 \cdot 0 - 1 + c} = \frac{9}{c-1} = 1 \quad \Longrightarrow \quad c = 10.$$

Hence, the particular solution to this IVP is

$$y(x) = \frac{9e^{3x}}{e^{3x}(3x-1)+10}.$$

(It is hard to tell, but 10 is a large enough choice of c such that, in this case, the interval of definition for this solution is the entire real line $(-\infty, \infty)$.) A plot of $y(x)$ is below if you're curious.