Math 2210Q-002/010

Exam II

Apr 19^{th}, 2019

Name: Key

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: ______________________
(full name)

Questions:	1	2	3	4	5	6	Bonus	**Total**
Points:	15	20	~~20~~ 15	~~15~~ 20	15	15	10	100
Score:	13	11	13	18	9	11	1	76

Average

Percentage
76

1. (15 points) (a) (5 points) Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 2 & 6 & 8 \end{bmatrix}$. Calculate the determinant of A.

$$|A| = \begin{vmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 2 & 6 & 8 \end{vmatrix} = \begin{vmatrix} -2 & -7 & -9 \\ 0 & -2 & -3 \\ 0 & -1 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & 7 & 9 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= -2(-1) = \boxed{2}$$

(b) (5 points) Let $B = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} -2 & 0 & -7 & -9 \\ 7 & 3^{+} & 6 & -1 \\ 2 & 0 & 5 & 6 \\ 2 & 0 & 6 & 8 \end{bmatrix} \end{array}$. Calculate the determinant of B using a cofactor expansion.

$$|B| = 3 \cdot |A| = 6$$

$$= -0 + 3\begin{vmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 2 & 6 & 8 \end{vmatrix} - 0 + 0$$

(c) (2 points) Using the properties of the determinant, calculate $\det 2A$.

$$\det 2A = 2^3 \cdot \det A = 8 \cdot 2 = \boxed{16}$$

(d) (2 points) Using the properties of the determinant, calculate $\det B^3$.

$$\det B^3 = (\det B)^3 = \boxed{(6)^3 = 216}$$

(e) (1 point) Is B invertible? Justify your answer.

Yes. $\det B \neq 0$.

2. (20 points) Consider $\mathbb{P}_2$ the vector space of all polynomials of degree at most 2.

(a) (5 points) Show that the set of polynomials $\{1+t^2, t-3t^2, 1+t-3t^2\}$ is linearly independent.

$p_1 \quad p_2 \quad p_3 \qquad S' = \{1, t, t^2\}$

$$\begin{bmatrix} [p_1]_{S'} & [p_2]_{S'} & [p_3]_{S'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

Since $[p_1]_{S'}, [p_2]_{S'}, [p_3]_{S'}$ lin. indep.

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Longrightarrow \{p_1, p_2, p_3\}$ is linearly independent.

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"No free variables"

"3 pivots for 3 columns"

"IMT" etc. $\longrightarrow$ $\Rightarrow$ linearly independent.

(b) (5 points) Let U be the collection of all polynomials of the form $p(t) = a + t^2$ for any a in $\mathbb{R}$. Is U a subspace of $\mathbb{P}_2$? If so, show U shows the three properties of a subspace. If not, show one of these properties fail.

1) $\vec{0}$ in U? No. $\vec{0}$ in $\mathbb{P}_2$ is $p_1(t) = 0 + 0t + 0t^2 = 0$

$\vec{p}$ in $U \Rightarrow p(t) = a + t^2 \neq 0$ for any a.

$1t^2 + 0t^2$ is the issue

(c) (5 points) Let V be the collection of all polynomials $p(t)$ such that $p(1) = 0$. Is U a subspace of $\mathbb{P}_2$? If so, show U shows the three properties of a subspace. If not, show one of these properties fail.

1) $\vec{0}$ in U? Yes if $p(t) = 0$ then $p(1) = 0$.

2) p, q in $U \Rightarrow p(1) = 0$, $q(1) = 0$

$$(p+q)(1) = p(1) + q(1) = 0 + 0 = 0. \text{ So}$$

$p+q$ in U. ✓

3) p in $U \Rightarrow p(1) = 0$, $(cp)(1) = c \cdot p(1)$

$= c \cdot 0 = 0$

so cp in U.

U is a subspace. ✓

(d) (5 points) Define a transformation $T : \mathbb{P}^2 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(1) \\ p(1) \end{bmatrix}$. Show that T is linear. (That is, for any polynomials p, q and scalars c we have $T(p + q) = T(p) + T(q)$ and $T(cp) = cT(p)$.)

$$T(p+q) = \begin{bmatrix} (p+q)(1) \\ (p+q)(1) \end{bmatrix} = \begin{bmatrix} p(1) + q(1) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(1) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(1) \\ q(1) \end{bmatrix}$$

$$= T(p) + T(q)$$

$$T(cp) = \begin{bmatrix} (cp)(1) \\ (cp)(1) \end{bmatrix} = \begin{bmatrix} c(p(1)) \\ c(p(1)) \end{bmatrix} = c \begin{bmatrix} p(1) \\ p(1) \end{bmatrix} = cT(p).$$

3. (15 points) The following matrices are row equivalent:

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) For what k is Col(A) a subspace of $\mathbb{R}^k$? For what k is Nul(A) a subspace of $\mathbb{R}^k$?

A is 3×4 so Col(A) in $\mathbb{R}^3$ & k

and Nul(A) in $\mathbb{R}^4$.

(b) (5 points) Find a basis for Col(A).

Use pivot columns of A as indicated by B

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$$

(c) (5 points) Find a basis for Nul(A).

$\vec{x}$ in Nul(A) if $\vec{x} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$ using $[A\ \vec{0}]$ ~ $[B\ \vec{0}]$

So $\left\{ \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$.

4. (20 points) (a) (5 points) Let $A = \begin{bmatrix} 5 & 2 \\ 7 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Solve the equation $A\mathbf{x} = \mathbf{b}$ using Cramer's rule. No credit will be given for other methods of solution.

$$|A| = \begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} = -9$$

$$|A_1(\vec{b})| = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$|A_2(\vec{b})| = \begin{vmatrix} 5 & 3 \\ 7 & 1 \end{vmatrix} = -16$$

$$\Rightarrow \quad A\vec{x} = \vec{b} \text{ if } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_1 = \frac{|A_1(\vec{b})|}{|A|}, \quad x_2 = \frac{|A_2(\vec{b})|}{|A|}$$

So $\vec{x} = \begin{bmatrix} -1/9 \\ 16/9 \end{bmatrix}$.

(b) (5 points) Observe that $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. Give the coordinate vector for $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ from above relative to $\mathcal{B}$.

From above $x_1 \begin{bmatrix} 5 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ if $x_1 = -1/9$, $x_2 = 16/9$. So

$$[\vec{b}]_{\mathcal{B}} = \begin{bmatrix} -1/9 \\ 16/9 \end{bmatrix}$$

(c) (5 points) Compute the area of the parallelogram S with vertices $(0, -2), (5, 5), (2, -1), (7, 6)$.

$+(0,2) \to (0,0), (5,7), (2,1), (7,9)$

$$\text{Area } S = \left| \det \begin{bmatrix} 5 & 2 \\ 7 & 1 \end{bmatrix} \right| = |-9| = \boxed{9}$$

(d) (5 points) Compute the area of the image of the paralleogram $T(S)$ under the transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 3 & 7 \\ 1 & 1 \end{bmatrix}$ from above.

$$\text{area } T(S) = ||A|| \cdot \text{area } S$$

$$= |-4| \cdot 9 = \boxed{36}$$

5. (15 points) Let $D_{2\times 2}$ be the collection of all 2×2 diagonal matrices. That is

$$D_{2\times 2} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}.$$

With addition and scalar multiplication defined in the usual way for 2×2 matrices, show that $D_{2\times 2}$ verifies axioms 2, 4 and 9 of the vector space axioms.

(a) (5 points) Axiom 2: for all vectors $\mathbf{u}, \mathbf{v}$, we have $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

$\vec{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \vec{v} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ Notice $\vec{u}, \vec{v}$ in $D_{2\times 2}$ are arbitrary

$$\vec{u} + \vec{v} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} = \begin{bmatrix} c+a & 0 \\ 0 & d+b \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \vec{v} + \vec{u}.$$

(b) (5 points) Axiom 4: there is a zero vector $\mathbf{0}$ in V such that for any vector $\mathbf{u}$, we have $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in $D_{2\times 2}$. Note

$$\vec{u} + \vec{0} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \vec{u}$$

for any $\vec{u}$.

(c) (5 points) Axiom 9: for any vector $\mathbf{u}$ and any scalars c, d we have $c(d\mathbf{u}) = (cd)\mathbf{u}$.

Here c, d are arbitrary

$$c(d\vec{u}) = c\left(d\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = c\begin{bmatrix} da & 0 \\ 0 & db \end{bmatrix} = \begin{bmatrix} cda & 0 \\ 0 & cdb \end{bmatrix}$$

$$= \begin{bmatrix} (cd)a & 0 \\ 0 & (cd)b \end{bmatrix} = (cd)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$= (cd)\vec{u}.$$

6. (15 points) Indicate whether each statement is true or false by circling **True** or **False** appropriately.

(a) (3 points) $\det(A+B) = \det(A) + \det(B)$.

True **False**

(b) (3 points) If f is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and $f(t) = 0$ for some t, then f is the zero vector in V.

True **False**

(c) (3 points) The kernel of a linear transformation is a vector space.

True **False**

(d) (3 points) If $H = \mathrm{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

True **False**

(e) (3 points) If $P_{\mathcal{B}}$ is the change of coordinates matrix for $\mathcal{B}$, then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}\mathbf{x}$ for all $\mathbf{x}$ in V.

True **False**

7. (**Bonus:** 5 points) Assume A is invertible. Prove that $\det(A^{-1}) = (\det(A))^{-1}$.

(Note: By $(\det(A))^{-1}$, we mean $\frac{1}{\det(A)}$.)

$$1 = \det(I) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A^{-1})$$

✓

8. (**Bonus:** 5 points) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation from question 2d. That is

$$T(p) = \begin{bmatrix} p(1) \\ p(1) \end{bmatrix}.$$

Give bases for the kernel of T and the range of T.

Range $(T) = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} : \text{any } a \text{ in } \mathbb{R} \right\}$ so basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

p in kernel(T) if $T(p) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow p(1) = 0$.

~~Any~~ So p is $t-1$ or t^2-1 or some combination of these too

$\Rightarrow$ Basis is

$$\left\{ t-1, t^2-1 \right\}.$$