Math 2210Q-002/010 | # Exam I | **Mar 9^{th}, 2019**

Name: Key

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

Academic Integrity Agreement

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: ______________________
(full name)

Questions:	1	2	3	4	5	6	Bonus	**Total**
Points:	20	15	10	10	10	10	10	75
Score:								

Percentage

1. (20 points) Consider the following linear system of equations

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 3 \\ 4x_1 - x_2 + 6x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 5 \end{aligned}$$

(a) (5 points) Write a matrix equation and a vector equation which are equivalent to this system.

$$\begin{bmatrix} 1 & -1 & 3 \\ 4 & -1 & 6 \\ 2 & -1 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} \quad \text{for } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix}.$$

(b) (5 points) Solve the system of equations and give your solution in parametric vector form.

$$\begin{bmatrix} 1 & -1 & 3 & 3 \\ 4 & -1 & 6 & 9 \\ 2 & -1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 3 & -6 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 3 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{aligned} x_1 &= 2 - x_3 \\ x_2 &= -1 + 2x_3 \\ x_3 &\ \text{free} \end{aligned}$$

so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}$

(c) (5 points) Give the solution of the associated homogeneous equation to this system.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 4 & -1 & 6 & 0 \\ 2 & -1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{so} \quad \boxed{\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}$$

(d) (5 points) Are the vectors $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$ linearly independent? If not, give a linear dependence relation among these vectors.

No, the equation $x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a nontrivial solution. From the previous part $x_3 = 1 \Rightarrow x_1 = -1$, $x_2 = 2$ so

$$-\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is a linear dependence relation.

2. (15 points) Consider the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$.

(a) (5 points) Circle each expression that is defined.

(i) $A + B$ (ii) $B - C$ (iii) $C + B^T$ (iv) $CA - A$

(v) AB (vi) BAC (vii) CA^T (viii) $(BC)^T$

(b) (5 points) Compute $C(B + C^T)$.

$$B + C^T = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 2 & 0 \end{bmatrix}$$

$$C(B + C^T) = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 7 \\ 0 & 2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 & 7 \\ 4 & -2 & 14 \\ 6 & 2 & 21 \end{bmatrix}}$$

(c) (5 points) Find A^{-1}, and use it to solve the three matrix equations

(i) $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (ii) $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (iii) $A\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{5-6} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \qquad A\vec{x} = \vec{b} \implies \vec{x} = A^{-1}\vec{b}.$$

i) $\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -5 \\ 2 \end{bmatrix}}$

ii) $\vec{x} = A^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cdots = \boxed{\begin{bmatrix} 3 \\ -1 \end{bmatrix}}$

iii) $\vec{x} = A^{-1}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \cdots = \boxed{\begin{bmatrix} 19 \\ -7 \end{bmatrix}}$

3. (10 points) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix}.$$

(a) (5 points) Find the standard matrix of T. That is, find a matrix A such that for all $\mathbf{x} \in \mathbb{R}^3$, we have $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}}$$

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

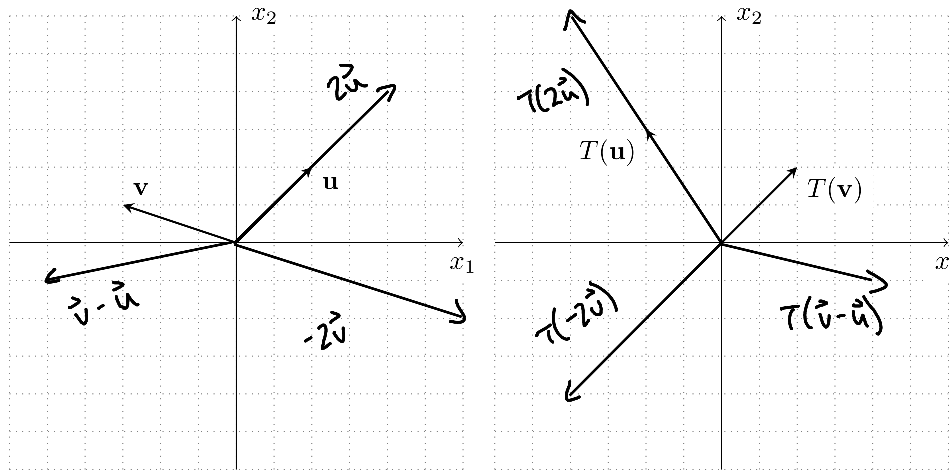
$$T(\vec{e}_3) = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

(b) (5 points) Is T one-to-one? Is T onto?

one-to-one? No, columns are linearly dependent (3 vectors each with 2 entries)

onto? Yes, pivot in each row.

4. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Consider the vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$ given below and to the right. Their images $T(\mathbf{u})$ and $T(\mathbf{v})$ are given below and to the left.



(a) (5 points) In the left plot, carefully sketch and label the vectors $2\mathbf{u}$, $-2\mathbf{v}$ and $\mathbf{v} - \mathbf{u}$.

(b) (5 points) In the right plot, carefully sketch and label the vectors $T(2\mathbf{u})$, $T(-2\mathbf{v})$ and $T(\mathbf{v} - \mathbf{u})$.

5. (10 points) Let H be an $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$. Suppose for a fixed $\mathbf{c}$ in $\mathbb{R}^n$ the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent.

(a) (5 points) Is H invertible? Justify your answer.

No, if H invertible, $H\vec{x} = \vec{c}$ has solution $\vec{x} = H^{-1}\vec{c}$ making the inconsistent equation $H\vec{x} = \vec{c}$ consistent.

(b) (5 points) Does the homogeneous question $H\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Justify your answer.

Yes, H is not invertible. If $H\vec{x} = \vec{0}$ has no nontrivial solution then H is invertible by the IMT.

6. (10 points) Indicate whether each statement is true or false by circling **True** or **False** appropriately.

(a) (2 points) Two matrices are row equivalent if they have the same number of rows.

True **False**

(b) (2 points) If a system is consistent and has free variables then the solution set contains infinitely many solutions.

True **False**

(c) (2 points) The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

True **False**

(d) (2 points) The columns of a matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

True **False**

(e) (2 points) If one row of an augmented matrix in echelon form is

$$\begin{bmatrix} 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

then the associated linear system must be inconsistent.

True **False**

7. (**Bonus:** 5 points) Show that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{bmatrix}$$

is *not* linear. (Hint: Consider e_1, e_2 and $e_1 + e_2$.)

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix}, \quad T(\vec{e}_1 + \vec{e}_2) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 5 \\ 5 \end{bmatrix}.$$

$$T(\vec{e}_1) + T(\vec{e}_2) = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 5 \end{bmatrix} \neq \begin{bmatrix} -1 \\ 5 \\ 5 \end{bmatrix} = T(\vec{e}_1 + \vec{e}_2).$$

So T not linear.

8. (**Bonus:** 5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Justify why $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ if

$$A = \begin{bmatrix} T(\mathbf{e_1}) & T(\mathbf{e_2}) & \cdots & T(\mathbf{e_n}) \end{bmatrix}$$

$\vec{x}$ in $\mathbb{R}^n \implies \vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$

T linear $\implies T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)$

$$= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A\vec{x}.$$