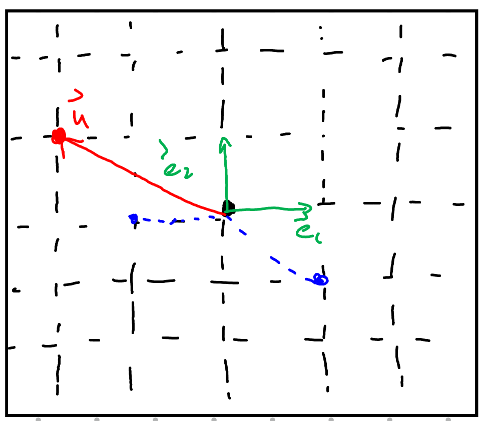
4.4: Coordinate systems

Key idea: Any basis describes each vector in a v.space by a unique linear combination. We consider the weights of this linear combination as the coordinates of the vector relative to this basis. Thus, different bases yield different coordinates and hence, different "views" of the space.

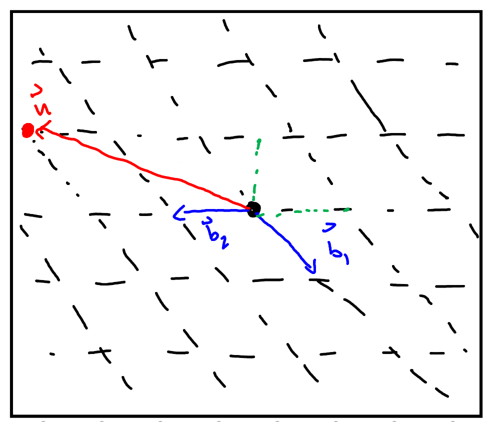
Ex| Consider the two bases $\mathcal{E}=\{\vec{e}_1,\vec{e}_2\}$ and $\mathcal{B}=\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}-1\\0\end{bmatrix}\right\}$ where $\vec{b}_1=\begin{bmatrix}1\\-1\end{bmatrix}$ and $\vec{b}_2=\begin{bmatrix}-1\\0\end{bmatrix}$. If $\vec{u}=\begin{bmatrix}-2\\1\end{bmatrix}$ then we see

Right board.

$$\vec{u}=-2\vec{e}_1+\vec{e}_2=-2\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad \vec{u}=-\vec{b}_1+\vec{b}_2=-\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}-1\\0\end{bmatrix}.$$



$\mathbb{R}^2$



$\mathbb{R}^2$

Observe that $\vec{u}$ is completely described by knowing which basis we are using and the weights needed to obtain $\vec{u}$ as a lin. comb. Different bases require different weights to describe a vector so we can think of them as giving different "views" of the vector space i.e. different paths to get to the same point.

These weights are unique for each vector so we make the following definition after stating this fact precisely:

Left board

Fact: If $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_n\}$ is a basis for a v.space V, then for each $\vec{v}$ in V there exist unique scalars $c_1,\dots,c_n$ s.t.

$$\vec{v}=c_1\vec{b}_1+c_2\vec{b}_2+\dots+c_n\vec{v}_n$$

Def: We call these unique scalars *the coordinates of $\vec{v}$ relative to $\mathcal{B}$* (or *$\mathcal{B}$-coordinates for $\vec{v}$*). The *coordinate vector of $\vec{v}$ relative to $\mathcal{B}$* (or *$\mathcal{B}$-coordinate vector*) is the vector from $\mathbb{R}^n$ with those scalars as entries. We write

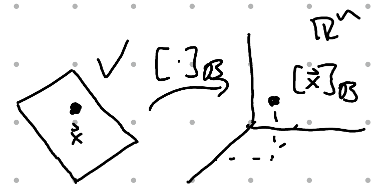
$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We'll see that this lets us "get a grip" on an arbitrary vector space by viewing it similar to $\mathbb{R}^n$

* later

We consider $[\cdot]_{\mathcal{B}}$ as a map from $V \to \mathbb{R}^n$ for each basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots \vec{b}_n\}$. Specifically

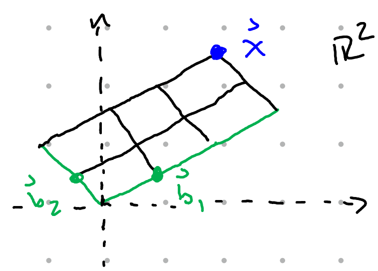
$$[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$$
$$\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$$



Ex| From above we see $[\vec{u}]_{\mathcal{E}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \vec{u}$ and $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(Notice for any $\vec{x}$ in $\mathbb{R}^n$, $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$ as $\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n$.)

Ex| Let $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. With $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, find the $\mathcal{B}$-coordinate vector of $\vec{x}$.



We require c_1, c_2 s.t. $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$.
That is,

$$c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \to c_1 = 3,\ c_2 = 2 \text{ so } [x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$[\vec{b}_1\ \vec{b}_2][x]_{\mathcal{B}} = \vec{x} \to P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x}$$

Def: For a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$, the matrix

$$P_{\mathcal{B}} = [\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_n]$$

is the *change of coordinates matrix* for $\mathcal{B}$.

In $\mathbb{R}^n$ $P_{\mathcal{B}}$ translates from $\mathcal{B}$-coordinates to standard coordinates

$$P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{E}} = \vec{x}$$

$$[\vec{x}]_{\mathcal{B}} \underset{P_{\mathcal{B}}^{-1}}{\overset{P_{\mathcal{B}}}{\rightleftarrows}} \vec{x} = [\vec{x}]_{\mathcal{E}}$$

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}[\vec{x}]_{\mathcal{E}} = P_{\mathcal{B}}^{-1}\vec{x}$$

Ex| (**) from above $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Indeed $P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ so $P_{\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

and $P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{E}}$ as well as

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = P_{\mathcal{B}}^{-1}[\vec{x}]_{\mathcal{E}}$.

so $P_{\mathcal{B}}^{-1}$ translates from standard coordinates into $\mathcal{B}$-coordinates.

(**) (Note: $P_{\mathcal{B}}$ has a basis of columns so it is always invertible $\Rightarrow P_{\mathcal{B}}: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, onto.)

We now turn to several examples to see how we may consider unfamiliar vectors as equivalent (and familiar) column vectors:

Ex| Let $S = \{1, t, t^2, t^3\}$, the standard basis of $\mathbb{P}_3$, and $\mathcal{B} = \{2, t-1, t^2, t^3+t\}$, a nonstandard basis of $\mathbb{P}_3$.

Find both S' and $\mathcal{B}$ coordinate vectors for $p = a + bt + ct^2 + dt^3$ in $\mathbb{P}_3$.

Clearly, $[p]_S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. To find $[p]_{\mathcal{B}}$, we proceed as above using the coordinate vectors of each element of $\mathcal{B}$.

$$c_1(2) + c_2(t-1) + c_3(t^2) + c_4(t^3+t) = a + bt + ct^2 + dt^3$$

$$\Big\downarrow [\cdot]_{S'}$$

$$c_1\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$P_{\mathcal{B}}[p]_{\mathcal{B}} = [p]_{S'}$$

To solve, row reduce:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2}(b-c-a) \\ 0 & 1 & 0 & 0 & b-d \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix} \Longrightarrow [p]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2}(b-c-a) \\ b-d \\ c \\ d \end{bmatrix}$$

e.g. $p = 1 + 2t + 3t^2 + 4t^3$, $q = 5 + t + 6t^3$

$\to [p]_{S'} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$

$\to [q]_{S'} = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 6 \end{bmatrix}$, $[q]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -5 \\ 0 \\ 6 \end{bmatrix}$

Ex Use the standard coordinate vectors to show $1+2t^2$, $4+t+5t^2$, $3+2t$ are linearly dependent in $\mathbb{P}_2$ and find a linear dependence relation among these vectors.

$$[1+2t^2]_S = \begin{bmatrix}1\\0\\2\end{bmatrix},\ [4+t+5t^2]_S = \begin{bmatrix}4\\1\\5\end{bmatrix},\ [3+2t]_S = \begin{bmatrix}3\\2\\0\end{bmatrix}$$

So we row reduce

$$\begin{bmatrix}1&4&3\\0&1&2\\2&5&0\end{bmatrix} \sim \begin{bmatrix}1&0&-5\\0&1&2\\0&0&0\end{bmatrix} \Rightarrow -5\begin{bmatrix}1\\0\\2\end{bmatrix}+2\begin{bmatrix}4\\1\\5\end{bmatrix}=\begin{bmatrix}3\\2\\0\end{bmatrix}$$

$$\Rightarrow \boxed{-5(1+2t^2)+2(4+t+5t^2)=3+2t}.$$

put this here

* We consider $[\cdot]_{\mathcal{B}}$ as a map from $V \to \mathbb{R}^n$ for each basis $\mathcal{B}=\{\vec{b}_1,\vec{b}_2,\dots\vec{b}_n\}$. Specifically

$[\cdot]_{\mathcal{B}}: V \to \mathbb{R}^n$

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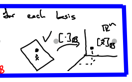

Notice the utility of the coordinate representation of a vector, (discussion of iso-morphism "same"-"form"), this allow us to treat all vector spaces as $\mathbb{R}^n$ and calculate in the simplest setting to study, every other setting in which a vector space is meaningful in math, science, or engineering.

Finally, we see a plane in $\mathbb{R}^3$ as "essentially" $\mathbb{R}^2$.

Ex Consider $H=\text{Span}\{\vec{v}_1,\vec{v}_2\}$ in $\mathbb{R}^3$ where $\vec{v}_1=\begin{bmatrix}3\\6\\2\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

Note $\mathcal{B}=\{\vec{v}_1,\vec{v}_2\}$ is a basis for H.

Determine if $\vec{x}=\begin{bmatrix}3\\12\\7\end{bmatrix}$ is in H and if so find $[\vec{x}]_{\mathcal{B}}$.

$\vec{x}$ in H if there are c_1, c_2 s.t. $c_1\vec{v}_1+c_2\vec{v}_2=\vec{x}$ (and then $[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$).

$$[\vec{v}_1\ \vec{v}_2\ \vec{x}] = \begin{bmatrix}3&-1&3\\6&0&12\\2&1&7\end{bmatrix} \sim \begin{bmatrix}1&0&2\\0&1&3\\0&0&0\end{bmatrix}\ \checkmark$$ so $\vec{x}$ in H and $[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}2\\3\end{bmatrix}$.

Notice that none of $\mathcal{E}=\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ are in H, so we need $\mathcal{B}$ to understand direction, orientation in a very natural subspace of $\mathbb{R}^3$: a plane, which $[\cdot]_{\mathcal{B}}$ renders indistinguishable from $\mathbb{R}^2$ ⟶

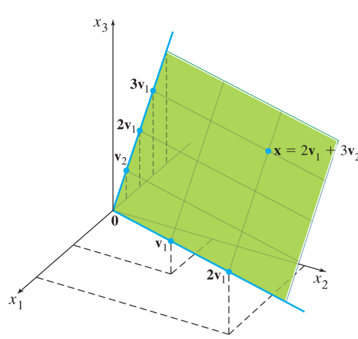

FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.