1.1: Systems of Linear equations

Wednesday, January 23, 2019 4:58 AM

Much of the core of linear algebra is motivated by, and exemplified in, systems of linear equations. This will become apparent as we move forward to understand the concepts of span, linear independence and linear transformations.

To begin, a linear equation is one that is linear in all variables:

Def A linear equation in the variables $x_1, \ldots, x_n$ is an equation which can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers.

Ex $x_1 + \sqrt{2}\,x_2 = x_3$ is linear

$\hookrightarrow$ $x_1 + \sqrt{2}\,x_2 - x_3 = 0$

$a_1 = 1$, $a_2 = \sqrt{2}$, $a_3 = -1$, $b = 0$

$x_1 + \sqrt{x_2} = x_3x_4$ is not linear.

The terms $\sqrt{x_2}$ and $x_3 x_4$ are not linear in the variables.

Def A system of linear equations (or linear system) is a collection of m linear equations in the same n variables $x_1, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots\cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad &\quad \vdots \\ a_{m1}x_1 + \cdots\cdots + a_{mn}x_n &= b_m \end{aligned}$$

Ex

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - 4x_3 &= -7. \end{aligned}$$

(2 equations in 3 variables)

Def A solution to a system of linear equations is collection $(s_1, s_2, \ldots, s_n)$ of n numbers that satisfy each equation when substituted for the n variables.

Ex1
$$2x_1 - x_2 + 1.5x_3 = 8$$
$$x_1 \qquad - 4x_3 = -7.$$

has a solution $(5, 6.5, 3)$ as

$$2(5) - (6.5) + 1.5(3) = 8$$

and

$$(5) \qquad - 4(3) = -7$$

are <u>both true</u>.

this system has infinitely many solutions in fact, e.g. $(-7, -22, 0)$ and $(-3, -12.5, 1)$.

Def1 The set of all solutions to a linear system is called a **solution set** and two systems are **equivalent** if they have the same soln set.

Geometrically we have two equivalent linear systems specify the same collection of points in $\mathbb{R}^n$.
In general, a solution set will consist of one, none, or infinitely many points. Put another way:

<u>Fact:</u> Linear systems have either
1) no solution.

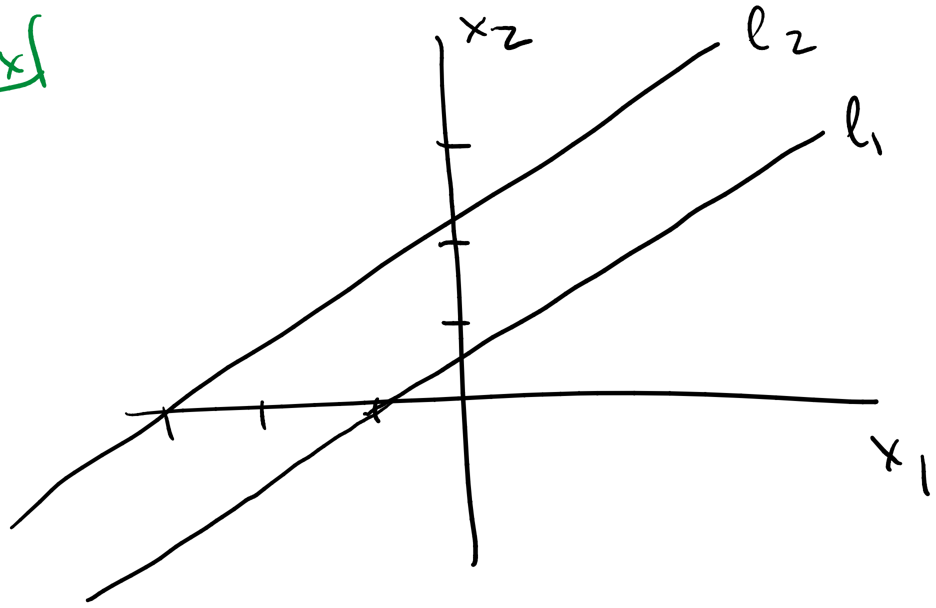
1) no solution,

2) exactly one solution,

3) or infinitely many solutions.

Def. A system is consistent if it has at least one solution, it is inconsistent if it has none.

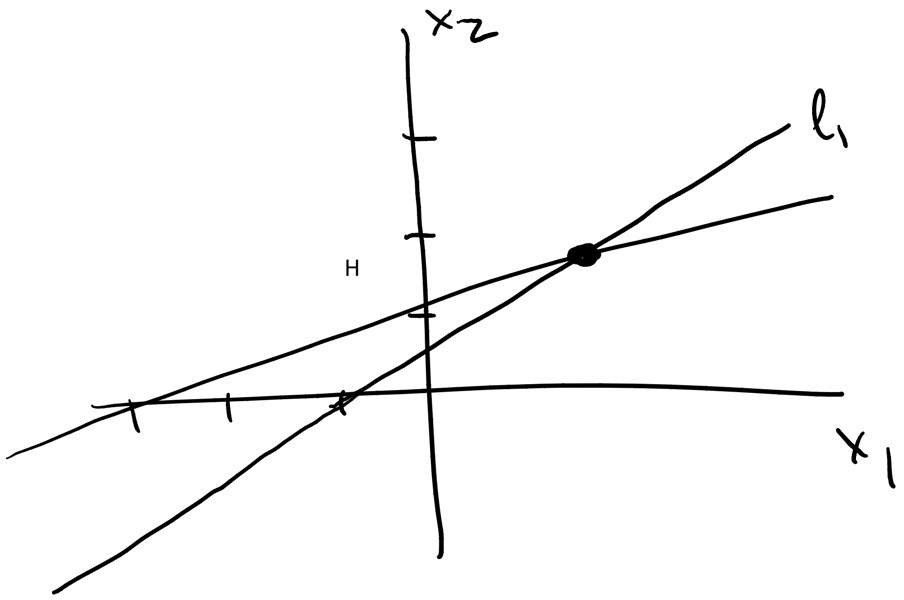
Ex/



(a) $x_1 - 2x_2 = -1 \quad (\ell_1)$

$-x_1 + 2x_2 = 3 \quad (\ell_2)$

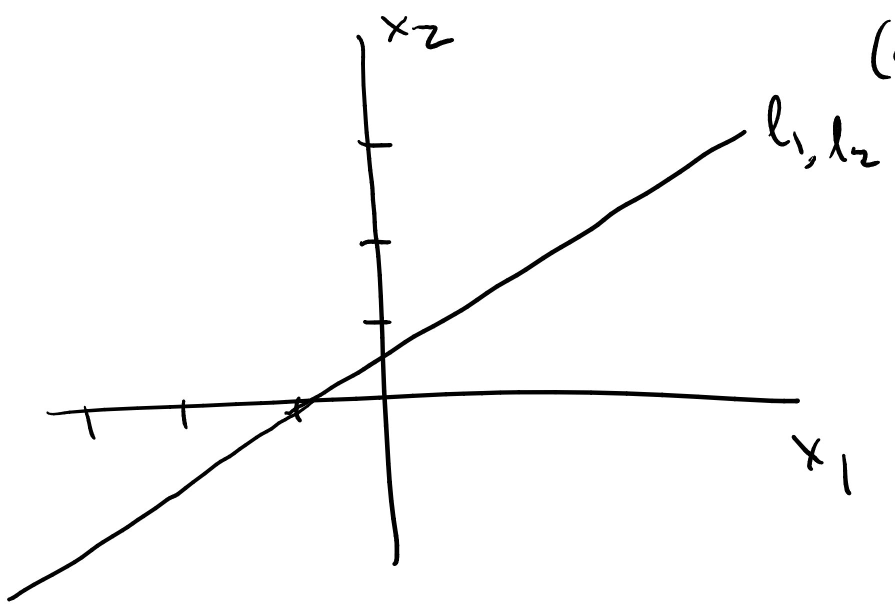
has no solution.



(b) $x_1 - 2x_2 = -1 \quad (\ell_1)$

$-x_1 + 3x_2 = 3 \quad (\ell_2)$

has exactly one solution.



(c) $x_1 - 2x_2 = -1 \quad (\ell_1)$

$-x_1 + 2x_2 = 1 \quad (\ell_2)$

has infinitely many solutions.

A similar situation occurs in $\mathbb{R}^3$ with 3-variable equations describing planes. (and further up to $\mathbb{R}^n$ and beyond!)

— Solving linear systems ——— made it to here —

You may have previously seen methods for solving systems of equations ("2 or 3 equations in 2 or 3 unknowns, etc.")

e.g. substitution.

Here, and next time, we describe an easily automated algorithm for solving systems of linear equations.

Ex | $x_1 - 2x_2 + x_3 = 0$

$2x_2 - 8x_3 = 8$

$5x_1 \qquad - 5x_3 = 10$

Note: 1) multiplying an equation by a scalar;
2) reordering equations;
3) adding equations together;
all preserve the solution set of a system

$-5 \cdot eq1$

$-5x_1 + 10x_2 - 5x_3 = 0$

$2x_2 - 8x_3 = 8$

$5x_1 \qquad - 5x_3 = 10$

With this in mind we use x_1 in equation 1 to eliminate x_1 from equation 3.

eq3 + eq1

$-5x_1 + 10x_2 - 5x_3 = 0$

$2x_2 - 8x_3 = 8$

We then can use x_2 in equation 2 to eliminate x_2 from

$$2x_2 \quad 6x_3 = 6$$

$-\frac{1}{5}$ eq1, $\frac{1}{2}$ eq2, $\frac{1}{10}$ eq3

$$10x_2 - 10x_3 = 10$$

eliminate x_2 from equations 1 and 3.

(first, simplify)

$$x_1 - x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$x_2 - x_3 = 10$$

eq1 + eq2, eq3 − eq2

(we now could find x_3 and substitute but let's continue)

$$x_1 \quad -3x_3 = 4$$
$$x_2 - 4x_3 = 4$$
$$3x_3 = 6$$

$\frac{1}{3}$ eq3

$$x_1 \quad -3x_3 = 4$$
$$x_2 - 4x_3 = 4$$
$$x_3 = 2$$

and now eliminate x_3 from equations 1 and 2

eq$_1$ + 3eq3, eq$_2$ + 4eq3

$$\boxed{\begin{aligned} x_1 &= 10 \\ x_2 &= 12 \\ x_3 &= 2 \end{aligned}}$$

And that is the solution to the system: geometrically, the single point of intersection of three planes.

Matrix Notation (later, formal take)

A good deal of redundant information was carried through that calculation, we abbreviate using matrices:

For the system we just solved

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned}$$

we associate a coefficient matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{pmatrix}$$

and an augmented matrix

$$\begin{array}{c} R_1 \to \\ R_2 \to \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right)$$

Now manipulating the equations amounts to manipulating the rows of this matrix.

eg) First steps of last calculation:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right) \xRightarrow{R_3 - 5R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right)$$

and so on

row equivalent

if one matrix can be row op.'d into another

row operations
1 replacement (algebra)
2 interchange
3 scaling

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Inconsistent systems

To determine if a system does not have a solution, we follow our algorithm until we find no way to proceed:

Ex1 Determine if

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$4x_1 - 8x_2 + 12x_3 = 1$$

is a consistent system.

We manipulate the rows of the augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right) \xrightarrow[R_1, R_2]{\text{interchange}} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_1} \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3 + 2R_2]{R_1 + 3R_2} \begin{pmatrix} 2 & 0 & 10 & 25 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 15 \end{pmatrix} \quad !$$

This translates to $0 = 15$.

No choice of x_1, x_2, x_3 can ever satisfy this,

thus the system is <u>inconsistent</u>.