Introduction

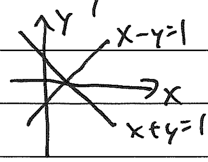
Date No. 1

'Define' linear algebra:

- for us, "linear" will mean flat (not curved)
- and "algebra" is the study of symbolic systems

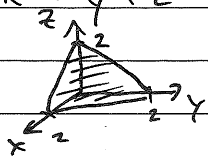
So in this course we study the symbolic representations of 'flat things'.
e.g. a line or plane

lines $\begin{cases} x+y=1 \\ x-y=1 \end{cases}$ are both satisfied by $x=1$, $y=0$.



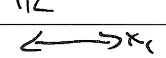
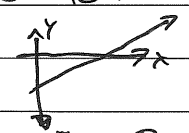
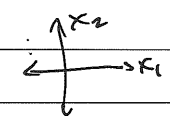
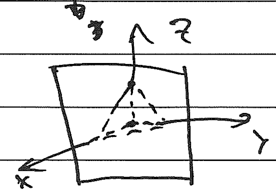
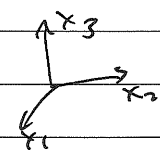
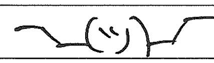
This symbollicaly represents the intersection of the two lines.

plane $\{ x+y+z=2$ is an algebraic representation of a plane.



we learn how to solve easy questions about these 'flat' objects

Now this may look a bit unfamillier if you have not seen multivariable calculus but this is about all you will have missed.

Linear object	equation	sketch	ambient space	
				$\mathbb{R}$ 
line	$x-2y=1$		$\mathbb{R}^2$	
plane	$2x+3y+2z=6$		$\mathbb{R}^3$	
hyperplane	$2x+y-z+4w=12$		$\mathbb{R}^4$	'tesseract'
	$2x_1+3x_2+\cdots+6x_7+(-3)x_8=0$		$\mathbb{R}^8$	

In general, we visualize the objects we study in <u>n-dimensional (Euclidean) space</u> $\mathbb{R}^n$.

The fundamental ~~concepts~~ objects of the course:

<u>Linear equations</u>: $x+y=2$, $\begin{cases} x_1 - x_2 + x_3 = 4 \\ x_1 \quad\quad - x_3 = 0 \\ \quad x_2 \quad\quad = 1 \end{cases}$ ← system

<u>Matrices</u>: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{14} \\ \vdots & & \ddots & \vdots \\ b_{51} & b_{52} & \cdots & b_{54} \end{bmatrix} = [b_{ij}]$

<u>Vectors</u>: In physics: ↗ ← ↙

In multivariable calc: $\langle 1 \;\; 3 \;\; 17 \rangle$

-or- $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

In linear algebra: abstract notion of a vector $\vec{v}$.

We will use these to work at our fundamental goal:

understanding the equation $\boxed{A\vec{x} = \vec{b}}$

at first: shorthand for systems of linear equations

later as matrix equations

finally as a linear transformation between two vector spaces.

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To conclude: three disclaimers for the course:

1) Linear algebra is ridden with vocabulary.
   - we will use words precisely and many of them, we will avoid imprecise terms like "vector form"
   - study the vocab often

2) Linear algebra is difficult to learn (and extremely useful) because it is simple.
   - we will learn one computational method by the end of the week (called row reduction)
   - we will spend the subsequent 14 weeks studying what we can learn from this computation (codifying what we learn in <u>the invertible matrix theorem</u>)

3) Linear algebra is elegant, and that's easy to miss.
   - remember to step back and ask yourself what connections to the broad picture of the course you can make while studying the specifics.
   - we're going to spend a lot of times looking at trees, but the forest is where the magic lies.

1.1 Systems of Linear Equations

A central motivation for the material and a chief application of the techniques there-in relate to linear systems.

<u>Def</u>: linear equation - - - $\qquad 6x+3z=-2y$ is linear

<u>Ex</u>| $\underbrace{\overset{\uparrow}{\underset{a_1}{\,}}x_1 + \sqrt{2}\,\underset{a_2}{x_2}}_{\text{coefficients}} = \underset{b}{3}$, $\quad \underbrace{\underset{a_1}{6}x + \underset{a_2}{2}y + \underset{a_3}{3}z}_{\text{coefficients}} = \underset{b}{0}$

<u>Non-ex</u>| $\sin(x)=1 \qquad x_1+\sqrt{x_2} = x_3x_4$

<u>Def</u>: system of linear equations - - -

$x_1 = 3 - x_3;\ 4, 5$
$x_2 = -x_3;\ 1, 2$
$x_3 = ;\ -1, -2$
e.g. sol'ns

<u>Ex</u>| $\begin{aligned} x_1 - x_2 &= 3 \\ x_1 + x_2 + 2x_3 &= 3 \end{aligned}$ is a system of 2 equations in 3 variables.

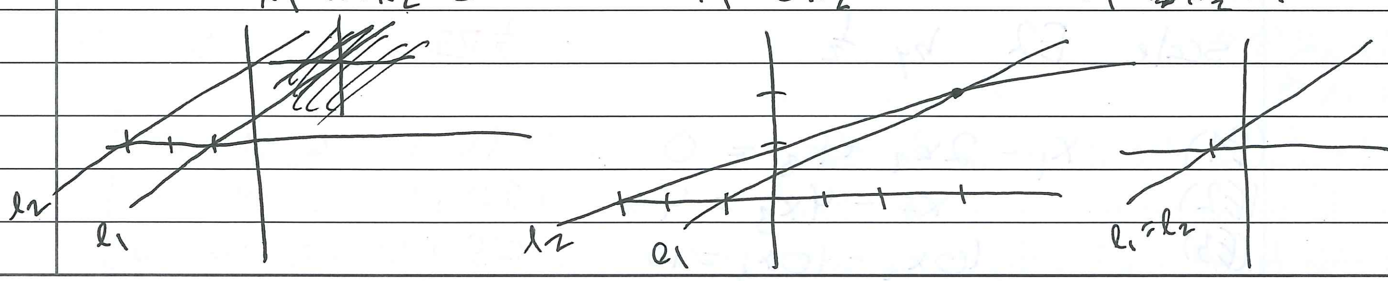
<u>Def</u>: solutions - - - (set, equivalent, consistent, inconsistent)

↳ motivates two important questions: given a lin. system

Existence: is the system consistent? or does at least one solution <u>exist</u>?

Uniqueness: if so, is there only one solution? -or- is the solution <u>unique</u>?

<u>Ex</u>| (a) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$ $\quad$ (b) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$ $\quad$ (c) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$



So (a) is inconsistent, consistency in 3-dim link
(b) is consistent with a unique solution
(c) is consistent with infinitely many solutions.

Solving linear systems: answering existence and uniqueness questions "or" finding a solution.

Every linear system has either
1) no solution;
2) a unique solution;
3) or infinitely many solutions

We now demonstrate how to ~~solve~~ ~~this~~ determine which case holds for a given linear system.

Def: coeff., augmented matrix, elementary operations, then

Ex1 Find all solutions of (goal: find an equivalent system with easy to read sol'n set)

$$\begin{array}{ll} (E1) & x_1 - 2x_2 + x_3 = 0 \\ (E2) & 2x_2 - 8x_3 = 8 \\ (E3) & 5x_1 - 5x_3 = 10 \end{array} \qquad \begin{array}{l} (R1) \\ (R2) \\ (R3) \end{array}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

replace E3 by E3 − 5E1 $\qquad R3: R3 - 5R1 \quad \sim$ ← "~~equivalent~~" or 'similar'

$$\begin{array}{ll} (E1) & x_1 - 2x_2 + x_3 = 0 \\ (E2) & 2x_2 - 8x_3 = 8 \\ (E3) & 10x_2 - 10x_3 = 10 \end{array} \qquad \begin{array}{l} (R1) \\ (R2) \\ (R3) \end{array}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

scale E2 by $\frac{1}{2}$ $\qquad \frac{1}{2}R2 \quad \sim$

$$\begin{array}{ll} (E1) & x_1 - 2x_2 + x_3 = 0 \\ (E2) & x_2 - 4x_3 = 4 \\ (E3) & 10x_2 - 10x_3 = 10 \end{array} \qquad \begin{array}{l} (R1) \\ (R2) \\ (R3) \end{array}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

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replace E3 by E3 − 10E2   R3: R3 − 10R2 S

(E1) $x_1 - 2x_2 + x_3 = 0$
(E2) $x_2 - 4x_3 = 4$
(E3) $30x_3 = -30$

$$\begin{array}{c} (R1) \\ (R2) \\ (R3) \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

scale E3 by 1/30 $\frac{1}{30}$ R3 S

* echelon form

(E1) $x_1 - 2x_2 + x_3 = 0$
(E2) $x_2 - 4x_3 = 4$
(E3) $x_3 = -1$

$$\begin{array}{c} (R1) \\ (R2) \\ (R3) \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(notice we may solve by back substitution: we however continue towards the simplest equivalent system)

for an example of inconsistency
(E1) $x_1 - 2x_2 + x_3 = 0$
(E2) $2x_2 - 8x_3 = 8$
(E3) $5x_1 \quad -35x_3 = 10$

replace E2 by E2 + 4E3   R2: R2 + 4R3 S
and E1 by E1 − E3   R1: R1 − R3

(E1) $x_1 - 2x_2 = 1$
(E2) $x_2 = 0$
(E3) $x_3 = -1$

$$\begin{array}{c} (R1) \\ (R2) \\ (R3) \end{array} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

* reduced echelon form

replace E1 by E1 + 2E2   R1: R1 + 2R2 S

(E1) $x_1 = 1$
(E2) $x_2 = 0$
(E3) $x_3 = -1$

$$\begin{array}{c} (R1) \\ (R2) \\ (R3) \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$A\vec{x} = \vec{b}$
$\Rightarrow (EA)\vec{x} = E\vec{b}$
$(EA)\vec{x} = E\vec{b}$
$\Rightarrow E^{-1}EA\vec{x} = E^{-1}E\vec{b}$
$\Rightarrow A\vec{x} = \vec{b}$ ✓

as the systems are equivalent, $x_1 = 1$, $x_2 = 0$, $x_3 = -1$ is the only solution to the original system.

Note: it may be hard to see why replacement works: this is a good question and will be much easier to justify after a touch of matrix theory.

1.2 Row reduction and echelon forms

As any linear system is represented by an augmented matrix, we give an algorithm to "row reduce" any matrix to a form (namely [reduced] echelon form) from which the solution set of the corresponding system can be deduced immediately.

Def: leading entry, (reduced) echelon form...

Ex1
$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -35 & 10 \end{bmatrix}$$

(forward phase)

row-reduction

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & -30 \end{bmatrix} \overset{\text{(back phase)}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the reduced echelon form $\qquad$ an echelon form

Fact: uniqueness of the reduced echelon form $\quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Def: pivots, pivot column, pivot position (example above).

Def: Row reduction algorithm, forward, backward phase..

Ex1 Find an echelon form and the reduced echelon form of:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

pivot position / 3 is a pivot

$$\begin{bmatrix} \boxed{0} & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \overset{R1 \leftrightarrow R3}{\sim} \begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

pivot column

$$R2 : R2 - R1 \quad \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

next pivot position

pivot column

to clear the 3 below 2, we opt to simply the calculation by next doing $\frac{1}{2}R2$

1 is always a nice pivot

$$\tfrac{1}{2}R2 \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \quad R3 : R3 - 3R2 \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This completes the forward phase and yields an echelon form of our original matrix. $\longrightarrow$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

We complete the backward phase to find the reduced echelon form.

$$\begin{matrix} R2 : R2 - R3 \\ \sim \\ R1 : R2 - 6R3 \end{matrix} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R1 : R1 + 9R2 \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -\cancel{63}72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\tfrac{1}{3}R1 \sim \underbrace{\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}}$$

And we have found the reduced echelon form.

Solution sets of linear systems via echelon form

The solution set of a system is readily found from the reduced echelon form. (of the augmented matrix)

pivot columns ≈ basic variables, non-pivot columns ≈ free variables

Ex:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 = 1$	$x_1 - 2x_3 + 3x_4 = -24$	$x_1 + x_2 = 0$
$x_2 = 0$	$x_2 - 2x_3 + 2x_4 = -7$	$x_3 = 0$
$x_3 = -1$	$x_5 = 4$	$0 = 1$
Unique solution $(1, 0, -1)$	Inf. many sol'ns $\{(s_1, s_2, s_3, s_4, s_5) :$ any value of s_3, s_4 where $s_1 = -24 + 2s_3 - 3s_4$, $s_2 = -7 + 2s_3 - 2s_4$, $s_5 = 4\}$	no solution

Def: basic and free variables, parametric description of ...

Ex:

$x_1 = 1$	$x_1 = -24 + 2x_3 - 3x_4$	no solution
$x_2 = 0$	$x_2 = -7 + 2x_3 - 2x_4$	
$x_3 = -1$	$x_5 = 4$	
	x_3, x_4 free	

Fact: existence and uniqueness w.r.t. echelon form.

1.3 Vector Equations

Here we meet the first form of the final fundamental object in the course: the vector.
We first consider these objects algebraically and then visualize them geometrically.

The Algebra of (column) vectors.

Def: column vectors, zero vector, $\vec{x}$ in $\mathbb{R}^n$

e.g. $\vec{u}$, $\vec{v}$, $\vec{b}$, $\vec{x}$, $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 1 \\ 14 \\ 3 \\ 28 \end{bmatrix}$

Def: algebraic operations

$2\vec{b} + \vec{c} = \vec{d}$

$\vec{c} = \vec{d} - 2\vec{b}$

e.g. $\vec{u} \neq \vec{v}$, $\vec{u} + \vec{v}$, $\vec{b} + \vec{c}$, $2\vec{u}$, $0\vec{v}$, $-\vec{c}$, $\vec{b} - \vec{c}$, $\cancel{2\vec{c} = \vec{d}}$, $\cancel{\vec{c} = \tfrac{1}{2}\vec{d}}$

Algebraic properties of (column) vectors in $\mathbb{R}^n$

- vector addition is 1) commutative, 2) associative, 3) $\vec{0}$ is the additive identity and 4) $-\vec{u}$ is the additive inverse of $\vec{u}$.
- scalar multiplication distributes over 1) vector addition and 2) scalar addition and is 3) associative.
  4) $c = 1$ is the scalar multiplicative identity.

Def: linear combinations

e.g. $2\vec{b} + \vec{c}$, $\frac{1}{3}\vec{b} - \vec{c} + (16)\vec{d}$, $\vec{u} + 2\vec{v}$, $3\vec{u}$

Ex) Determine whether $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ is a lin. comb. of $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$.

Equivalently, determine if there are scalars x_1, x_2 s.t.

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b} \longrightarrow \underbrace{x_1\begin{bmatrix}1\\-2\\-5\end{bmatrix} + x_2\begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}}_{\text{vector equation.}}$$

So we need solve this vector equation.

$$x_1\begin{bmatrix}1\\-2\\-5\end{bmatrix} + x_2\begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix} \rightarrow \begin{bmatrix}x_1\\-2x_1\\-5x_1\end{bmatrix} + \begin{bmatrix}2x_2\\5x_2\\6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix}$$

$$\begin{bmatrix}x_1+2x_2\\-2x_1+5x_2\\-5x_1+6x_2\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix} \rightsquigarrow \begin{aligned} x_1+2x_2&=7\\ -2x_1+5x_2&=4\\ -5x_1+6x_2&=-3\end{aligned}$$ a linear system

$$[\vec{a}_1\ \vec{a}_2\ \vec{b}] = \begin{bmatrix}1&2&7\\-2&5&4\\-5&6&3\end{bmatrix} \sim \begin{bmatrix}1&2&7\\0&9&18\\0&16&32\end{bmatrix} \sim \begin{bmatrix}1&2&7\\0&1&2\\0&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}$$

which has solution $x_1 = 3$, $x_2 = 2$ $\Longrightarrow 3\vec{a}_1 + 2\vec{a}_2 = \vec{b}$. (Check!)

So yes, $\vec{b}$ is a lin. comb. of $\vec{a}_1, \vec{a}_2$ with weights $x_1 = 3$, $x_2 = 2$.

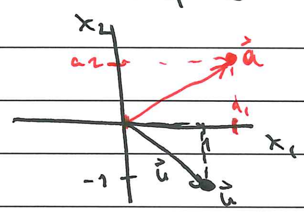
<u>Fact</u>: vector equations and linear equations. motivates the study of

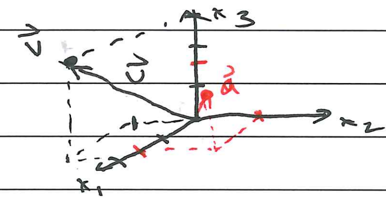
Def: Span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$

<u>The Geometry of vectors</u>

<u>"Def"</u>: visualizing vectors in $\mathbb{R}^n$

e.g. $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$; $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$

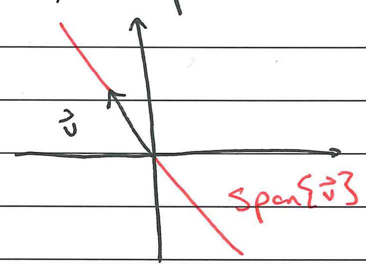


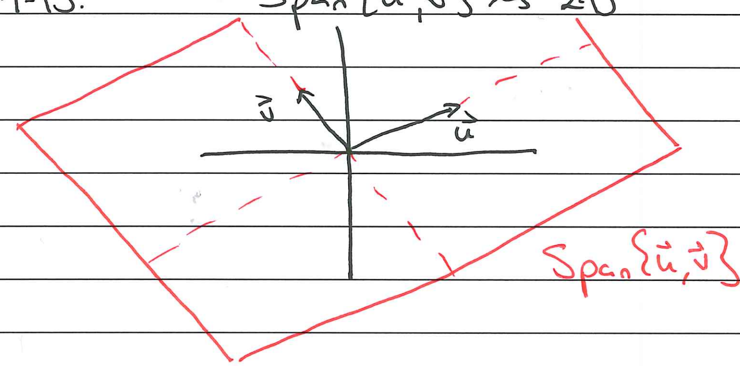


Visualizing addition and scaler multiplication . . .

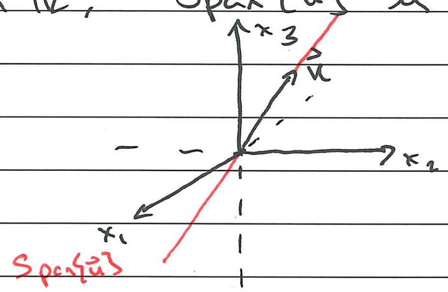
Visualizing span:

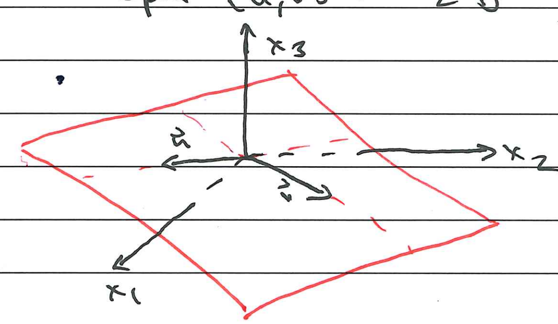
in $\mathbb{R}^2$, Span$\{\vec{v}\}$ is 1-D. Span$\{\vec{u}, \vec{v}\}$ is 2D



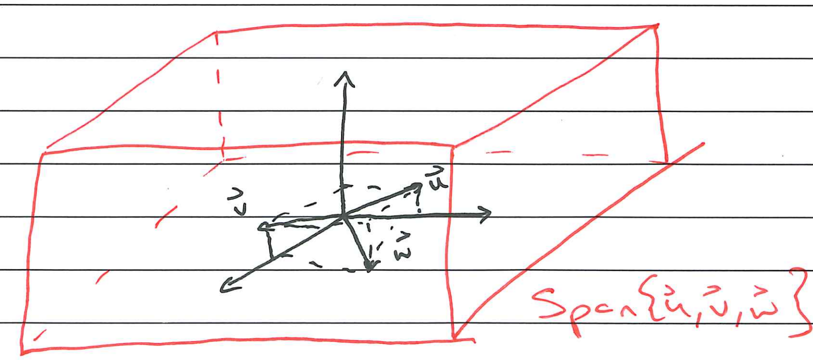


in $\mathbb{R}^3$, Span$\{\vec{u}\}$ is 1-D, Span$\{\vec{u}, \vec{v}\}$ is 2-D





Span$\{\vec{u}, \vec{v}, \vec{w}\}$
is 3-D



1.4 The matrix equation $A\vec{x}=\vec{b}$

We now formally connect the matrices of §1.2 to the equations in §1.1 and §1.3.

Def: $A\vec{x} = \underbrace{\begin{bmatrix}\vec{a}_1 & \cdots & \vec{a}_n\end{bmatrix}}_{m\times n}\underbrace{\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}}_{n\times 1} = \underbrace{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n}_{\text{a lin. comb. of col. of } A \text{ w/ } \vec{x} \text{ entries as weights}}$

Ex/ $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Ex/ $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\cancel{\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$, $\vec{b} = \begin{bmatrix} 2 \\ -12 \\ -8 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & 3 & 4 \\ \boxed{-4 \quad 2 \quad -6} \\ -3 & -2 & -7 \end{bmatrix}\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = x_1\begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix}$$

row-vector rule

$$= \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ \boxed{-4x_1 + 2x_2 - 6x_3} \\ -3x_1 - 2x_2 - 7x_3 \end{bmatrix}$$

Note: two properties arise from this definition...

Consider $A\vec{x} = \vec{b}$: $\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ -8 \end{bmatrix}$

matrix eq'n

$$x_1\begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ -4x_1 + 2x_2 - 6x_3 \\ -3x_1 - 2x_2 - 7x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ -8 \end{bmatrix}$$

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 2 \\ -4x_1 + 2x_2 - 6x_3 = -12 \\ -3x_1 - 2x_2 - 7x_3 = -8 \end{cases}$$

Fact: all share solution set.

Is $A\vec{x}=\vec{b}_*$ consistent? Yes. ~~Solve system by row reduc~~

Row reduce $[A \ \vec{b}] = \begin{bmatrix} 1 & 3 & 4 & 2 \\ -4 & -2 & -6 & -12 \\ -3 & -2 & -7 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

So $x_1 = 1$, $x_2 = -1$, $x_3 = 1$ solves the corresponding system,

$\Longrightarrow \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ solves $A\vec{x} = \vec{b}_*$.

To check this: compute $A \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and make sure it is $\vec{b}_* = \begin{bmatrix} 2 \\ -12 \\ -8 \end{bmatrix}$

Note: The equation $A\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A. (i.e. $\vec{b}$ is in the span of the columns of A)

Let's answer the broader question of whether or not $A\vec{x}=\vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^3$.

Ex| Is $A\vec{x}=\vec{b}$ consistent for any $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$?

Row reduce the augmented matrix $[A \ \vec{b}]$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 7 & 5 & b_2+3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2+4b_1 \\ 0 & 0 & 0 & \underbrace{b_3+3b_1-\tfrac{1}{2}(b_2+4b_1)}_{=b_1-\frac{1}{2}b_2+b_3} \end{bmatrix}$$

So $A\vec{x}=\vec{b}$ is consistent so long as $b_1 - \frac{1}{2}b_2 + b_3 = 0$. e.g. $A\vec{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not.

The issue is the lack of a pivot in row 3. See last fact.

1.5 Solution Sets of Linear Systems

We now utalize our understanding of the equation $A\vec{x}=\vec{b}$ to gain a greater understanding of the solution sets of linear systems both algebraically and geometrically.

We begin with a study of the solution sets of a simple class of linear systems:

Def: homogeneous systems, trivial solution e.g. $A_1\vec{x}=\vec{0}$ and $A(\vec{0})=\vec{0}$.

Ex/ We solve the three homogeneous linear systems

$$A_1\vec{x}=\vec{0} \qquad A_2\vec{x}=\vec{0} \qquad A_3\vec{x}=\vec{0}$$

Solution set:

$$A_1=\begin{bmatrix}3 & 5 & -4\\ -3 & -2 & 4\\ 6 & 1 & -8\end{bmatrix} \underset{\text{(row reduction)}}{\sim} \begin{bmatrix}1 & 0 & -4/3\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad \begin{array}{l} x_1 = 4/3x_3\\ x_2=0\\ x_3 \text{ free}\end{array}$$

$$A_2=\begin{bmatrix}1 & -3 & -2\\ -1 & 3 & 2\\ 2 & -6 & -4\end{bmatrix} \overset{''}{\sim} \begin{bmatrix}1 & -3 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad \begin{array}{l} x_1=3x_2+2x_3\\ x_2,x_3 \text{ free}\end{array}$$

$$A_3=\begin{bmatrix}1 & 5 & -3\\ 0 & -3 & 2\\ 3 & -3 & 1\end{bmatrix} \overset{''}{\sim} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \begin{array}{l} x_1=0\\ x_2=0\\ x_3=0.\end{array}$$

- Notice this is all we need to understand reduced echelon form of $[A_1\ \vec{0}]$, $[A_2\ \vec{0}]$, and $[A_3\ \vec{0}]$.

- Notice: ~~fact~~ nontrivial solution iff free variable.

Solutions in vector form:

~~$A_1\vec{x}=\vec{0}$ if $\vec{x}=x_3\begin{bmatrix}4/3\end{bmatrix}$~~

Consider the solution sets in vector forms:

$A_1\vec{x}=\vec{0}$ only if $\vec{x}=\begin{bmatrix}4/3\,x_3\\0\\x_3\end{bmatrix} = x_3\begin{bmatrix}4/3\\0\\1\end{bmatrix}$

for any x_3 in $\mathbb{R}$

$A_2\vec{x}=0$ only if $\vec{x}=\begin{bmatrix}3x_2+2x_3\\x_2\\x_3\end{bmatrix} = \underbrace{x_2\begin{bmatrix}3\\1\\0\end{bmatrix}+x_3\begin{bmatrix}2\\0\\1\end{bmatrix}}$

for any x_2, x_3 in $\mathbb{R}$

$A_3\vec{x}=\vec{0}$ only if $\vec{x}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\vec{0}$

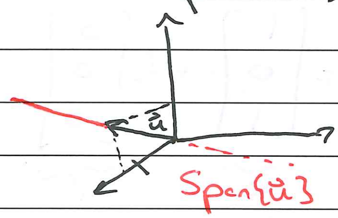
parametric vector form (x_3; x_2, x_3 as parameters)

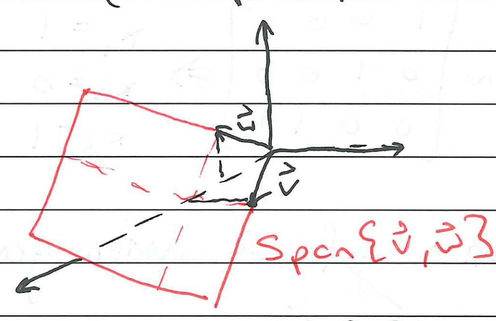
Geometrically: let $\vec{u}=\begin{bmatrix}4/3\\0\\1\end{bmatrix}$, $\vec{v}=\begin{bmatrix}3\\1\\0\end{bmatrix}$, $\vec{w}=\begin{bmatrix}2\\0\\1\end{bmatrix}$; r, s, t scalars

solution set of: (these are the parametric vector forms of a line and plane)

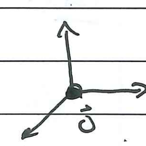
$A_1\vec{x}=\vec{0}$ is $\vec{x}=r\vec{u}$ for any r. (or $\mathrm{Span}\{\vec{u}\}$)

$A_2\vec{x}=\vec{0}$ is $\vec{x}=s\vec{v}+t\vec{w}$ for any s, t. (or $\mathrm{Span}\{\vec{v},\vec{w}\}$)





Note: solution set of $A_3\vec{x}=\vec{0}$ is $\{\vec{0}\}$ so



Nonhomogenous systems: $A\vec{x}=\vec{b}$ with $\vec{b}\neq\vec{0}$.

Notice if $\vec{p}$ is a particular solution to $A\vec{x}=\vec{b}$ and $\vec{u}$ is a homogeneous solution then $\vec{p}+\vec{u}$ is <u>also</u> a solution to $A\vec{x}=\vec{b}$

$$A(\vec{p}+\vec{u}) = A\vec{p}+A\vec{u} = \vec{b}+\vec{0}=\vec{b} \checkmark$$

Ex| Give in parametric vector form the solution sets of

$$A_1\vec{x} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} \qquad A_2\vec{x} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \qquad A_3\vec{x} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$$

Row reduce

$$\left[\begin{array}{c|c} & 7 \\ A_1 & -1 \\ & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & 8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

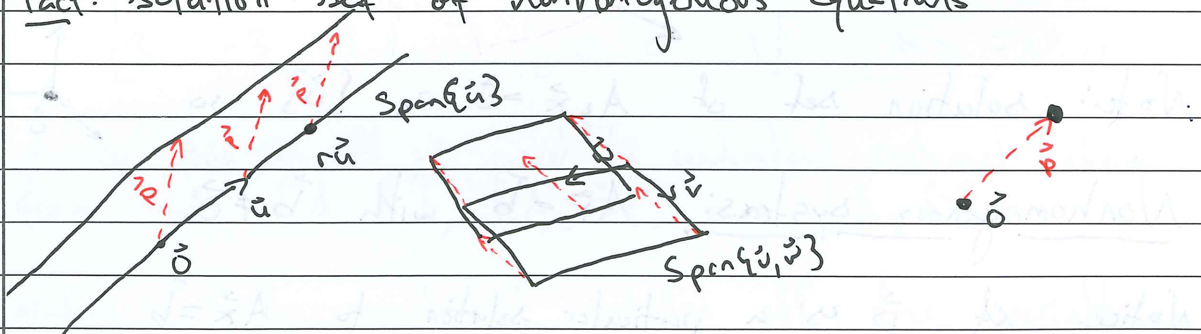
notice is same as before

$$\Rightarrow \begin{array}{l} x_1 = -1 + 4/3\, x_3 \\ x_2 = 2 \\ x_3 \text{ is free} \end{array} \quad \text{so} \quad \vec{x} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{u} \text{ from before}}$$

$$\left[\begin{array}{c|c} & 2 \\ A_2 & -2 \\ & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + x_2 \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{u}}$$

$$\left[\begin{array}{c|c} & -2 \\ A_3 & 2 \\ & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = 0 \\ x_3 = 1 \end{array} \Rightarrow \vec{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{p}} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

Fact: solution set of nonhomogeneous equations ...



Geometrically, the solution set of $A\vec{x} = \vec{b}$ is the shifted set of the homogeneous equation.

1.7 Linear independence

We isolate the notion of linear indepence to precisify our discussion of span and thereby improve our understanding of the solution sets of linear systems $A\vec{x}=\vec{b}$.

Def: linear independence . . . .

Ex) Determine if $\vec{v}_1=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{v}_3=\begin{bmatrix}2\\1\\0\end{bmatrix}$ are lin. indep.

If so $x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3=\vec{0}$ has only the trivial solution. To see if this is the case, we row reduce $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{0}]$

$$[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{0}]=\begin{bmatrix}1&4&2&0\\2&5&1&0\\3&6&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&0\end{bmatrix}\Rightarrow \begin{array}{l}x_1=2x_3\\x_2=-x_3\ ;\ x_3 \text{ free}\end{array}$$

So, $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are linearly dependent! As there are nontrivial solutions to $x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3=\vec{0}$

Choose $x_3=1 \Rightarrow x_1=2,\ x_2=-1$ and then

$2\vec{v}_1-\vec{v}_2+\vec{v}_3=\vec{0}$ is a linear dependence relation for $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$.

(Check this calculation is true ↑)

Ex) Are the columns of A linearly independent where

$A=\begin{bmatrix}0&1&4\\1&2&-1\\5&8&0\end{bmatrix}$? Yes if $x_1\begin{bmatrix}0\\1\\5\end{bmatrix}+x_2\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}+x_3\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}=\vec{0}$ has only the trivial solution, equivalently, if $A\vec{x}=\vec{0}$ has only the trivial solution. (Here $\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$)

$[A\ \vec{0}]\sim\begin{bmatrix}\boxed{1}&2&-1&0\\0&\boxed{1}&4&0\\0&0&\boxed{13}&0\end{bmatrix}$ All columns of A are pivots! So no free variable, so $A\vec{x}=\vec{0}$ has a unique solution ($\vec{x}=\vec{0}$) and thus, the columns of A are linearly independent.

Intuitively, $\{\vec{v}_1, \dots, \vec{v}_p\}$ are linearly indepent if no vector v_j is in the span of the others.

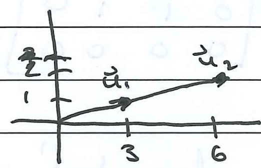
Ex| Determine if (i) $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

and (ii) $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ are lin. indep.

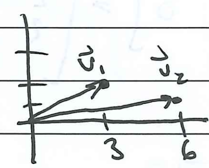
Note: two vectors are lin. dep. if $c\vec{v} + d\vec{w} = \vec{0}$ with $d \neq 0$
$\Rightarrow \vec{w} = \left(\frac{-c}{d}\right)\vec{v}$
$\Rightarrow \vec{w} \in \text{Span}\{\vec{v}\}$

i) $\vec{u}_2 = 2\vec{u}_1$ so $-2\vec{u}_1 + \vec{u}_2 = \vec{0}$ is a lin. dep. relation for $\vec{u}_1, \vec{u}_2$.

ii) $\vec{v}_1, \vec{v}_2$ are not multiples of one another so must be lin. independent.

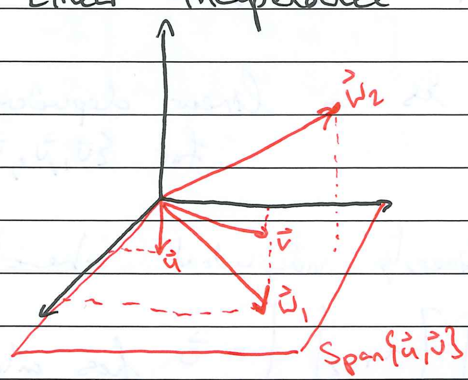
Geometrically: i) 

lin. dependent

ii) 

lin. independent

Ex| Linear independence in $\mathbb{R}^3$



$\vec{w}_1$ in Span$\{\vec{u},\vec{v}\}$ so $\{\vec{u},\vec{v},\vec{w}_1\}$ is lin. dependent

$\vec{w}_2$ is not in Span$\{\vec{u},\vec{v}\}$ so $\{\vec{u},\vec{v},\vec{w}_2\}$ is lin. independent.

($\vec{w}_1 \in \text{Span}\{\vec{u},\vec{v}\}$ so $a\vec{u} + b\vec{v} = \vec{w}_1 \Rightarrow a\vec{u} + b\vec{v} - \vec{w}_1 = \vec{0}$ is a lin dep. relation)

Note: two other tests for $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ to be lin. dep.

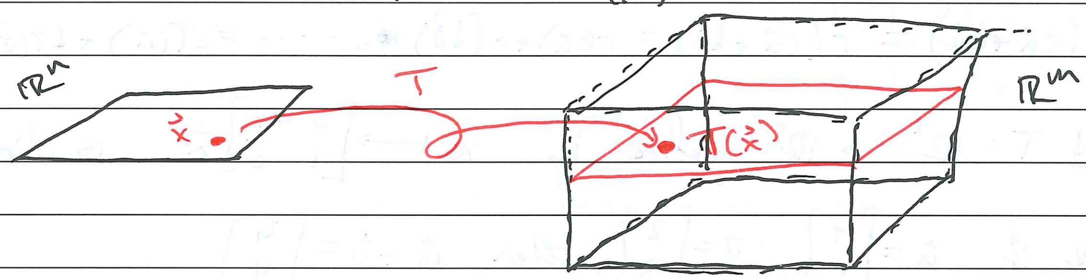
- $\vec{0}$ in S: $\vec{0}$ always in Span of other vectors
- $p > n$ for $v_1, \dots, v_p$ in $\mathbb{R}^n$: - always have a free variable or
  - geometrically can't escape the span of first n (lin. indep) vectors

1.8 Intro. to Linear Transformations

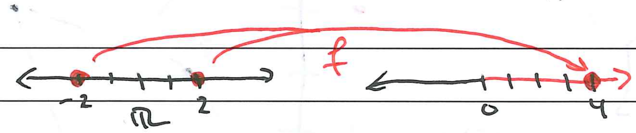
Now with a solid initial understanding of vectors in $\mathbb{R}^n$, we move to consider functions of vectors which we call transformations.

<u>Def</u>: transformations, domain, codomain, image, range . . .

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \longmapsto T(\vec{x})$$



These terms should be familiar from functions $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$



<u>Matrix transformations</u>: every $m \times n$ matrix A defines $T: \mathbb{R}^n \to \mathbb{R}^m$ by $\underset{\mathbb{R}^n}{\vec{x}} \longmapsto \underset{\mathbb{R}^m}{A\vec{x}}$

<u>Ex</u> Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 2 & 1 \end{bmatrix}$ and define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by
$\vec{x} \longmapsto A\vec{x}$.

So, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$ then

domain?
codomain?
range?

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

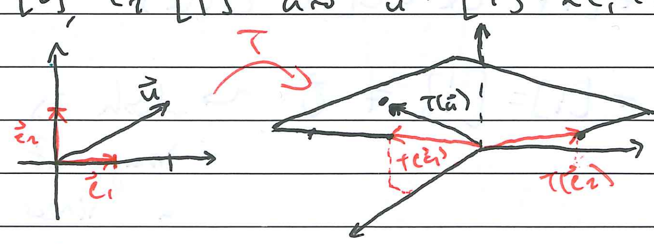
Consider $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{e}_1 + \vec{e}_2$

$T(\vec{e}_1) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

$T(\vec{e}_2) = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$

$T(\vec{u}) = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$

$= 2T(\vec{e}_1) + T(\vec{e}_2)$



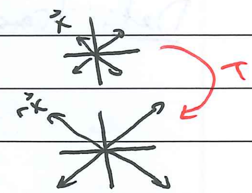
Def: Linear transformations . . . facts.

Ex Any matrix transformation is linear

$T(c\vec{u}+d\vec{v}) = A(c\vec{u}+d\vec{v}) = A(c\vec{u})+A(d\vec{v}) = c(A\vec{u})+d(A\vec{v}) = cT(\vec{u})+dT(\vec{v})$

Ex Dilation by r is a linear transformation.

$T(\vec{x}) = r\vec{x}$ for $\vec{x}$ in $\mathbb{R}^n$ $(T: \mathbb{R}^n \to \mathbb{R}^n)$



$T(c\vec{u}+d\vec{v}) = r(c\vec{u}+d\vec{v}) = r(c\vec{u})+r(d\vec{v}) = \cdots = cT(\vec{u})+dT(\vec{v})$

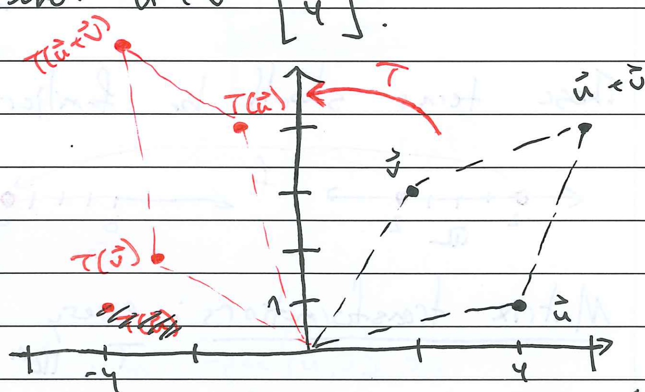
Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\vec{x} \longmapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$ is linear.

Note if $\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\vec{u}+\vec{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

$T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$T(\vec{v}) = \cdots = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$T(\vec{u}+\vec{v}) = \cdots = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$



Note: $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ Geometrically: T is a rotation $(\pi/2)$

Non-ex Show $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $\vec{x} \longmapsto \vec{x}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is *not* linear.

- $T(\vec{x}+\vec{y}) = \vec{x}+\vec{y}+\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \vec{x}+\vec{y}+\begin{bmatrix} 2 \\ 2 \end{bmatrix} = T(\vec{x})+T(\vec{y})$

or

- $T(c\vec{x}) = c\vec{x}+\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq c\vec{x}+c\begin{bmatrix} 1 \\ 1 \end{bmatrix} = cT(\vec{x})$

or

- $T(\vec{0}) = \vec{0}+\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \vec{0}$ $\leadsto$ which is always the image of $\vec{0}$ under a linear transformation.

1.9 The Matrix of a Linear Transformation

We now show that every linear transformation can be viewed as a matrix transformation using the matrix whose columns are the images of the identity matrix columns.

Def: Identity matrix $I_n = [\vec{e}_1 \ \vec{e}_2 \ \cdots \ \vec{e}_n]$

Note: any $\vec{x}$ in $\mathbb{R}^n$ is a lin. comb. of $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n = x_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

So if $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, we have

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)$$

Thus: the image of any $\vec{x}$ is a lin. comb. of the images of $\vec{e}_1, \dots, \vec{e}_n$

Key point

Ex1 $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear, $T(\vec{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$.

Find the image of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = T(2\vec{e}_1 - \vec{e}_2) = 2T(\vec{e}_1) - T(\vec{e}_2) = 2\begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -22 \\ 4 \end{bmatrix}$$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1\begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}.$$

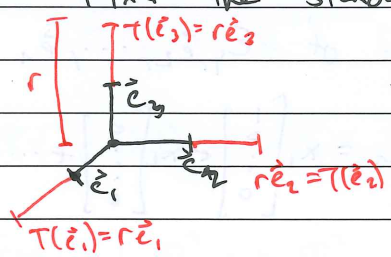
Notice:
$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [T(\vec{e}_1) \ \ T(\vec{e}_2)]\vec{x}.$$

Def: standard matrix and fact.

Note: $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \vec{x} = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)$.

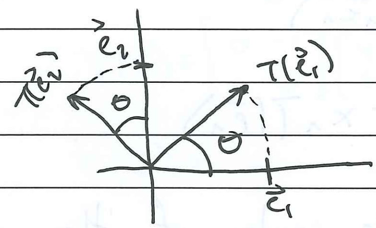
Ex| Fix r and let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the dilation $T(\vec{x}) = r\vec{x}$.
Find the standard matrix of T.



$T(\vec{e}_1) = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \cdots$ so $A = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$.

Check: $T(\vec{x}) = A\vec{x}$.

Ex| Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x})$ is the result of rotating $\vec{x}$ about the origin by θ radians. (ccw)



$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ so $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

Ex| Matrix multiplication is composition of linear transformations
e.g. standard matrix of reflect across $x_1 = x_2$ then ccw by $\pi/4$.

We consider a few more examples from pgs 74-76 of the text.

To conclude: the notions of one-to-one and onto for transformations

Ex| If $T: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation with standard matrix

$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

Is T onto? Yes. Pivot in every row so $A\vec{x} = \vec{b}$ always consistent.

Is T one-to-one? No. Free variable so ~~($T(\vec{0}) = \vec{0}$ only solution?)~~ nontrivial solution to $T(\vec{x}) = \vec{0}$, so columns lin. dependent.

2.1 Matrix Operations

We begin our study of matrices themselves with the basic definitions and algebra.

Def: Matrices, entries, diagonality, etc. . . .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Notice $A^T = [a_{ji}]_{\substack{1 \le j \le n \\ 1 \le i \le m}}$

$$= [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_j \cdots \vec{a}_n]$$

Def: algebra.

. . . .

- $A = B$ if $a_{ij} = b_{ij}$ for all i, j

- $A + B = [a_{ij} + b_{ij}]$ and $cA = [ca_{ij}]$

Ex] $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 & 4 \\ 3 & 3 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$3A, -2B$

Compute dimensions, a_{32}, c_{23}, $A+B$, $B+C$ undefined, $A+B = C^T$, $D^T = D$,

properties of the basic algebra

Matrix Multiplication: . . . $AB = [A\vec{b}_1 \ A\vec{b}_2 \cdots A\vec{b}_p]$

Ex] Let $A = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}}_{3\times 2}$, $B = \underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}}_{2\times 4}$. Compute AB if it is defined.

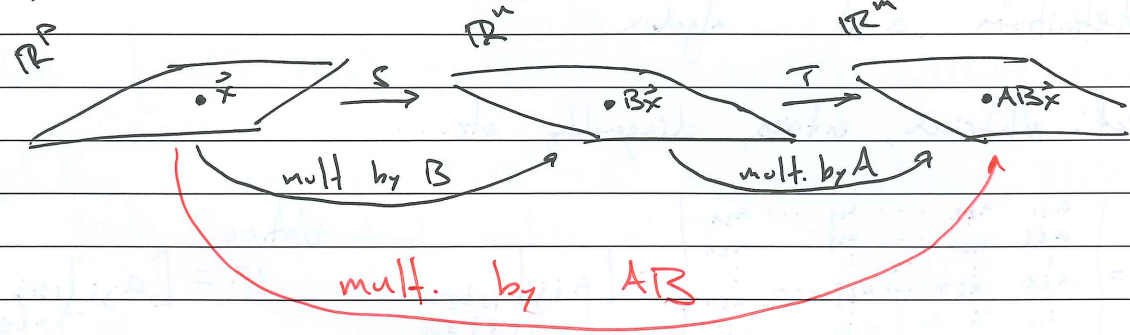
$$\underbrace{AB}_{3\times 4} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 8 & 0 & 2 \\ 2 & 4 & 4 & -1 \end{bmatrix}$$

$$\begin{array}{cccc} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 & A\vec{b}_4 \end{array}$$

$A\vec{b}_1 = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

$A\vec{b}_2 = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$, $A\vec{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $A\vec{b}_4 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

Why this definition? Composition of transformations



Ex] Compute (using the row-column rule) the products AB and BA for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -4 & 11 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

$\boxed{\underline{\text{Notice}} \quad AB \neq BA.}$

We now consider what properties do hold for matrix multiplication as well as those that do not.

Properties and warnings for matrix multiplication: - - - -

2.2 The Inverse of a Matrix

Key idea: Some matrices admit multiplicative inverses (in the sense that if A $n \times n$, $A \cdot A^{-1} = A^{-1} \cdot A = I_n$). We will see that these matrices are particularly nice in that invertibility is tied to many notions we are already concerned with. For now, we concern ourselves with finding the inverse of a particular square matrix A.

We've already seen that the addition, subtraction, and multiplication we are used to extends to matrices. Today we investigate an extension of division.

Recall division in the real numbers: $5 \cdot 5^{-1} = 5/5 = 1$ and $3 \cdot 5^{-1} = 3/5 = 0.6$
Dividing by 5 is equivalent to multiplying by it's multiplicative inverse $5^{-1} = 1/5$.
(Similarly, subtraction by 5 is equivalent to adding by it's additive inverse -5.)
So, our notion of "division by a matrix A" will be multiplying by it's multiplicative inverse A^{-1}, i.e., the matrix s.t.

$$A \cdot A^{-1} = I_n = A^{-1} A$$

Here A, A^{-1} are $n \times n$ and both AA^{-1} and $A^{-1}A$ are important b/c commutativity fails for matrices.

This matrix is unique but more on that later.

Ex) Let $A = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 & 1/5 \\ -1 & -1/2 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$.

Show $C = A^{-1}$ and $B \neq A^{-1}$.

$$AC = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix} = CA.$$

✓ so $AC = I_2$, $CA = I_2$ so $C = A^{-1}$.

$$BA = \begin{bmatrix} 1/2 & 1/5 \\ -1 & -1/2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0.8 & 2.1 \\ -1.5 & -4 \end{bmatrix}.$$

$BA \neq I_2$ so $B \neq A^{-1}$

So, we see inverting A is a subtle business, simply inverting all the elements of A failed to produce an inverse, which begs the question:

How do we find A^{-1} for a given A?

Fact: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

has an inverse

If $ad-bc=0$, then A is not invertible $\rightsquigarrow$ has no inverse.

We call the quantity $ad-bc$ the determinant of A, written $\det(A)$,
if $\det(A)=0$, A has no inverse and we sometimes call A a singular matrix,
if $\det(A)\neq 0$, A has an inverse, and we call it nonsingular.

Before describing why this is the inverse of a 2×2, we consider a familiar application.

Ex Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. Find A^{-1} and solve $A\vec{x}=\vec{b}$.

Note $ad-bc = 3\cdot6 - 4\cdot5 = -2$ so $A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$.

Now, $A\vec{x}=\vec{b}$: $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$: $\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

$\Rightarrow I_2\vec{x} = A^{-1}\vec{b}$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

$\Rightarrow \vec{x} = A^{-1}\vec{b}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

So matrix inverses allow the equation $A\vec{x}=\vec{b}$ to quickly be solved:

$$\vec{x} = A^{-1}\vec{b}.$$

Fact: If A is an invertible $n\times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x}=\vec{b}$ has a unique solution: $\vec{x} = A^{-1}\vec{b}$.

So matrix inverses greatly help us in manipulating matrix equations so, we need understand how to find them in general.

Given an $n \times n$ matrix A, we seek $A^{-1} = [\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n]$ s.t. $AA^{-1} = I_n$.
By the definition of matrix multiplication, we see:

$$AA^{-1} = [A\vec{x}_1 \ A\vec{x}_2 \cdots A\vec{x}_n] = [\vec{e}_1 \ \vec{e}_2 \cdots \vec{e}_n] = I_n .$$

So we need only solve the n equations: $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2 \cdots A\vec{x}_n = \vec{e}_n$.
And this can be done through row reduction!

Ex Find the inverse of $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ 2 & 6 & -5 \end{bmatrix}$.

We write $A^{-1} = [\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3]$ and solve

$$A\vec{x} = \vec{e}_1 \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & -7 & 3 & 0 \\ 2 & 6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -10 \end{bmatrix} \Longrightarrow \vec{x}_1 = \begin{bmatrix} -17 \\ -14 \\ -10 \end{bmatrix}$$

$$A\vec{x} = \vec{e}_2 \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & -7 & 3 & 1 \\ 2 & 6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$A\vec{x} = \vec{e}_3 \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & -7 & 3 & 0 \\ 2 & 6 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -17 & 4 & -1 \\ -14 & 3 & -1 \\ -10 & 2 & -1 \end{bmatrix}$.

Notice A is row equivalent to I and in fact, could have solve all three equations at once because $[A \ I] \sim [I \ A^{-1}]$.

Notice!! $[A \ I] = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -17 & 4 & -1 \\ 0 & 1 & 0 & -14 & 3 & -1 \\ 0 & 0 & 1 & -10 & 2 & -1 \end{bmatrix} = [I \ A^{-1}]$.

You should check all the calculations mentioned above, including that indeed, $A^{-1} \cdot A = A \cdot A^{-1} = I_3$ and $[A \ I] \sim [I \ A^{-1}]$.

The algorithm for finding A^{-1}: If A is a square matrix and invertible then $A \sim I$ and $[A \ I] \sim [I \ A^{-1}]$. Else A is not invertible

This works for any matrix! Let's consider an example from the webpage: work/talk through that quick maple worksheet.

To finish a few facts:

1) If A is invertible, A^{-1} is invertible and $(A^{-1})^{-1} = A$.

2) If A and B are invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1} \Rightarrow (AB)^{-1}(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$$

3) If A is invertible, so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Part 2 generalizes to any product of invertible matrices: the inverse is the product of the inverses in reverse order,

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}.$$

Next time we consider a fundamental result in linear algebra: the consequences and characterizations of invertibility.

2.3 The invertible Matrix theorem

Key idea: The invertible matrix theorem provides a fundamental connection between most of the concepts we've studied and the invertibility of a matrix. We can now answer broad questions about both systems of n linear equations in n variables and collections of n vectors from $\mathbb{R}^n$ by only knowing if a matrix is invertible.

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

This serves as a broad characterization of invertibility in many contexts.

This result is fundamental: consider it carefully in terms of all the concepts we've studied thus far.

Ex| Determine if $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible (or any of other properties listed above...)

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$ $\rightsquigarrow$ three pivots! Part (c) of the theorem guarantees A is invertible.

This is an important result that you will become familiar with via your homework. it simply takes practice and time considering it.

To conclude our discussion of invertibility we turn our attention to invertible transforms:

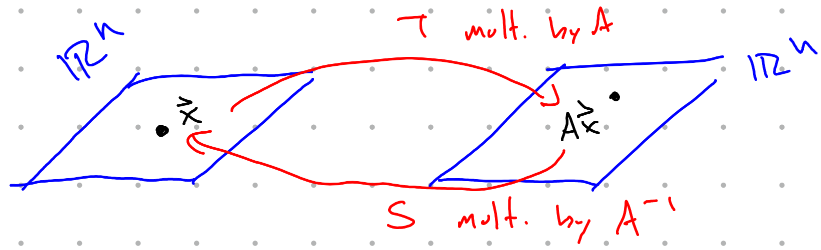
Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there is a function $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\text{for all } \vec{x} \text{ in } \mathbb{R}^n, \quad T(S(\vec{x})) = \vec{x} = S(T(\vec{x})).$$

Fact: If S exists it is unique and linear, so we may say S is <u>the</u> inverse of T.

T is invertible if and only if it's standard matrix A is invertible (thus the invertible matrix thm implies T is, for example, one-to-one and onto). Furthermore, the standard matrix of S, the inverse of T, is A^{-1} so $S(\vec{x}) = A^{-1}x$.

This fact should be intuitive: $T(S(\vec{x})) = A \cdot A^{-1}\vec{x} = I_n \vec{x} = \vec{x}$ and $S(T(\vec{x})) = A^{-1} \cdot A\vec{x} = I_n\vec{x} = \vec{x}$.



Ex) If $T: \mathbb{R}^n \to \mathbb{R}^n$ is onto, the columns of it's standard matrix A span $\mathbb{R}^n$. So by the invertible matrix theorem, A is invertible. In particular, item (e) says the columns of A are linearly independent so T <u>must</u> also be one-to-one.

(This argument works in reverse too: T one-to-one $\Rightarrow$ T onto.)

3.1 Determinants (Introduction)

Key idea: The notion of determinant we saw for a 2×2 matrix can be extended to any $n\times n$ matrix. Much like in 2×2 setting, $\det(A)$ tells us a lot about an $n\times n$ matrix A, namely if A is invertible or not. We will see in section 3.3 the geometric information $\det(A)$ gives about the transformation $\vec{x} \longmapsto A\vec{x}$.

Recall that a 2×2 matrix is invertible if it is row-equivalent to I_2, so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{a\neq 0}{\sim} \begin{bmatrix} a & d \\ 0 & d - bc/a \end{bmatrix} \sim \begin{bmatrix} a & 0 \\ 0 & ad-bc \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ so long as $ad-bc \neq 0$.

i.e., so long as $\det(A)\neq 0$.

We now extend the definition of $\det(A)$ to all $n\times n$ matrices and study this quantity in detail. (Note: the book (pg 166) gives an algebraic justification for higher dimensional determinants much like we did for 2×2. The algebra is a touch nasty, so we omit this.)

Def: Let $A=[a_{ij}]$ be an $n\times n$ matrix, we define *the determinant of A*, written $\det(A)$ as follows: if $n=1$, $\det(A) = \det([a]) = a$.

if $n\geq 2$, define A_{ij} to be the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A, then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^n a_{1n}\det(A_{1n})$$

Notice this is an alternating sum of determinants for smaller matrices. To compute $\det(A)$, we need first know $\det(A_{1j})$ and so on. In practice, this involves a lot of computation but we will see methods to make calculation much easier.

Ex Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$A_{11} = \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}$

$A_{12} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

$A_{13} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$

$$\det(A) = 1\cdot\det(A_{11}) - 5\det(A_{12}) + 0\det(A_{13})$$

So we need find A_{11}, A_{12}, A_{13} and their determinants

For brevity we write $|A|$ in place of $\det(A)$:

$$|A_{11}| = \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} = 0-2 = -2, \quad |A_{12}| = \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = 0-0=0, \quad |A_{13}| = \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = -4.$$

So $|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| = 1(-2) - 5(0) + 0(-4) = \boxed{-2}$

Notice how the presence of 0's in A simplify the calculation, the next fact allows us to fully exploit the structure of A to aid in finding determinants.

Def: For an $n\times n$ matrix A, the (i,j)-cofactor of A is: $\quad C_{ij} = (-1)^{i+j}\det(A_{ij})$.

Fact: The determinant of A is equal to the sum of the cofactors across any row or column, i.e.

for any i: $\quad \det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

and,

for any j: $\quad \det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$.

Note: We call such a sum a cofactor expansion across the i^{th} row (j^{th} column). The sum is always alternating with minus signs determined by a_{ij} position in A i.e.

$$\begin{array}{c} a_{11} \\ a_{12} \\ a_{13} \\ \vdots \end{array} \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \end{bmatrix}$$

This follows from the def. of C_{ij}:

$$(-1)^{i+j} = \begin{cases} +1 & i+j \text{ even} \\ -1 & i+j \text{ odd.} \end{cases}$$

Ex Recompute $\det A$ for

$$A = \begin{array}{c} \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \end{array} \begin{array}{ll} \leftarrow a_{31} & + \\ \leftarrow a_{32} & - \\ \leftarrow a_{33} & + \end{array}$$

using a cofactor expansion down the third column.

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (0)\overset{+1}{(-1)^{3+1}}\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} + (-1)\overset{-1}{(-1)^{3+2}}\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + (0)\overset{-1}{(-1)^{3+3}}\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

$$= 0 + (-2) + 0 = \boxed{-2} \checkmark$$

We conclude with another example of an informed choice of a cofactor expansion. We will see that triangular matrices are particularly nice to compute the determinant of.

Ex Compute $\det(B)$ where
$$B = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$
We choose a cofactor expansion involving many zeros: column 1.

$$\det(B) = 3C_{11} + 0C_{21} + 0C_{31} + 0C_{41} + 0C_{51}$$
$$= 3(-1)^{1+1}\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3\cdot\left(2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix} + 0\begin{vmatrix} \cdots \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix}\right)$$

another cofactor expansion

$$= 3\cdot 2\cdot\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$
$$= 3\cdot 2\cdot(-2) \quad \text{from above}$$
$$= \boxed{-12}.$$

Notice that as we expand down the first and second columns, we simply multiply a smaller determinant by the diagonal entries.
If every column of B was like the first two, the determinant would have simply been the product of the diagonal entries.

Fact: If A is a triangular matrix, then $\det A$ is the product of the diagonal entries of A.

On a numerical note: the number of calculations it takes to compute a determinant of a random $n\times n$ matrix is roughly $n!$ ... which is huge. To compute a 25×25 determinant requires 1.5×10^{25} operations. So, if we do a trillion operations a second, we'll need 500,000 years.

So how does Maple compute this in less than a second? There are some properties of $\det(A)$ that severely ease our computational task, and we heavily exploit that last fact.

3.2 Properties of Determinants

Key idea: If two matrices are row equivalent, then their determinants are related in precise ways. We mention these here and use them to compute determinants much more efficiently.

Recall that $\det(A)$ was very easy to compute in the case that A was triangular. Thus, if we knew how the determinant of a matrix was related to the determinant of an echelon form, the latter would be simple to compute making the former simple as well.
Indeed, we do know how these relate.

Write before lecture if possible

Fact: (The determinant and row reduction).
For A a square matrix

1) If a multiple of a row of A is added to another to produce a matrix B, then $\det(A) = \det(B)$.

2) If two rows of A are swapped to produce B then $\det(B) = -\det(A)$.

3) If one row of A is multiplied by k to produce B then $\det(B) = k \cdot \det(A)$

These properties make sense if you reconsider the definition of $\det(A)$ in light of them: e.g. prop 2 follows from rearranging where the cofactors are
$$(a_{1j} \leftrightarrow a_{2j} \Rightarrow C_{1j} \leftrightarrow -C_{2j}).$$

Let's now use these properties to compute some determinants:
Method: we reduce a matrix to echelon form, tracking how each operation affects the determinant. The matrix whose determinant we actually compute is triangular (so it's the product of the diagonal entries).

Ex Compute the determinant of $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

We "track" the reduction of A to make this an easier calculation

$$\det(A) = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = (-1)(1)(3)(-5) = \boxed{15}$$

combining rows doesn't affect the determinant

to get echelon form, we swap R2 and R3

det. of triangular matrices are simple!

Ex Compute $\det(A)$ if $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

$$\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

swap R1, R4 –or– Factor out 2

combining rows

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 2(1)(3)(-6)(1) = \boxed{-36}$$

Mult. by $\frac{1}{2}$ $\Rightarrow$ $\frac{1}{2}\det A = \det B$
so $\det A = 2\det B$

Thus we see in general that the determinant of A is linked closely to it's echelon form:

Fact: If A is reduced to U without scaling any rows and U is in echelon form then

note: this is always possible.

$$\det(A) = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{if } A \text{ invertible} \\ 0 & \text{if } A \text{ noninvertible} \end{cases}$$

A invertible $U = \begin{bmatrix} p & * & * \\ 0 & p & * \\ 0 & 0 & p \end{bmatrix}$

A noninvertible $U = \begin{bmatrix} p & * & * \\ 0 & p & * \\ 0 & 0 & 0 \end{bmatrix}$

On a numerical note: recall an $n \times n$ determinant takes $n!$ operations to compute via the definition. The method described here requires much less at $\frac{2}{3}n^3$ many operations. Thus, we need 10,000 operations for a 25×25 matrix compared to 1.5×10^{25} many. (less than a second vs 500,000 years.)

This method also leads to an important characterization of invertibility:

Fact: A is invertible if and only if $\det(A) \neq 0$.

Recall section 2.2 and 2.3 matrix inverses.

(So we gain an additional item in the invertible matrix theorem.)

To conclude, two final properties of the determinant:

Fact: 1) If A is $n \times n$, $\det(A^T) = \det(A)$

→ so all the facts about row operations hold about column operations

e.g. $\det[a_1\ a_2] = -\det[a_2\ a_1]$.

2) If A, B are $n \times n$, $\det(AB) = \det(A)\det(B)$.

Ex/ $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 2 & 7 \\ 8 & 8 \end{bmatrix} \rightsquigarrow \det(AB) = 16 - 56 = -40$

$\det(A)\det(B) = -10 \cdot 4 = -40$

$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$\det(A) = 2 - 12 = -10 \qquad \det(B) = 4 - 0 = 4$

NOTE: $\det(A+B) \neq \det(A) + \det(B)$ in general.

If you're curious, the determinant defines a (multi)linear transformation for every matrix A. This fact is studied in deeper analyses of the determinant. (One can show in some sense that the determinant is the only function that can do this.)

3.3 Cramer's rule & the geometry of the determinant

Key idea: After discussing a useful tool, we turn to understanding what a determinant tells us geometrically. We will see that the determinant gives us the factor by which a 1x1 square's area changes under the transformation determined by the matrix in question.

Before discussing the geometric information inherit in the determinant we briefly overview a useful theoretic tool: Cramer's rule.

Fix an $n \times n$ matrix A and for any $\vec{b}$ in $\mathbb{R}^n$ define a new matrix

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{b} & \cdots & \vec{a}_n \end{bmatrix}$$

↳ column i

by replacing the ith column of A with $\vec{b}$. Then the following is true:

Cramer's rule: If A is an invertible $n \times n$ matrix and $\vec{b}$ in $\mathbb{R}^n$, then the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries of the form

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}, \quad i = 1, \ldots, n.$$

We will see an example of this below but for high dimension, Cramer's rule is woefully inefficient at calculating solutions. It is useful theoretically though, as from Cramer's rule we can derive a formula (not algorithm) for the inverse of A. (See pg 181)

Ex) Use Cramer's rule to solve
$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$
so $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

So $A_1(\vec{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$, $A_2(\vec{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$ and by Cramer's rule:

$$x_1 = \frac{\det A_1(\vec{b})}{\det(A)} = \frac{24+16}{12-10} = \frac{40}{2} = 20.$$

$$x_2 = \frac{\det A_2(\vec{b})}{\det(A)} = \frac{24+30}{2} = \frac{54}{2} = 27$$

so $x_1 = 20$ and $x_2 = 27$ is the solution to the system.

"Now for the focus of this section:"

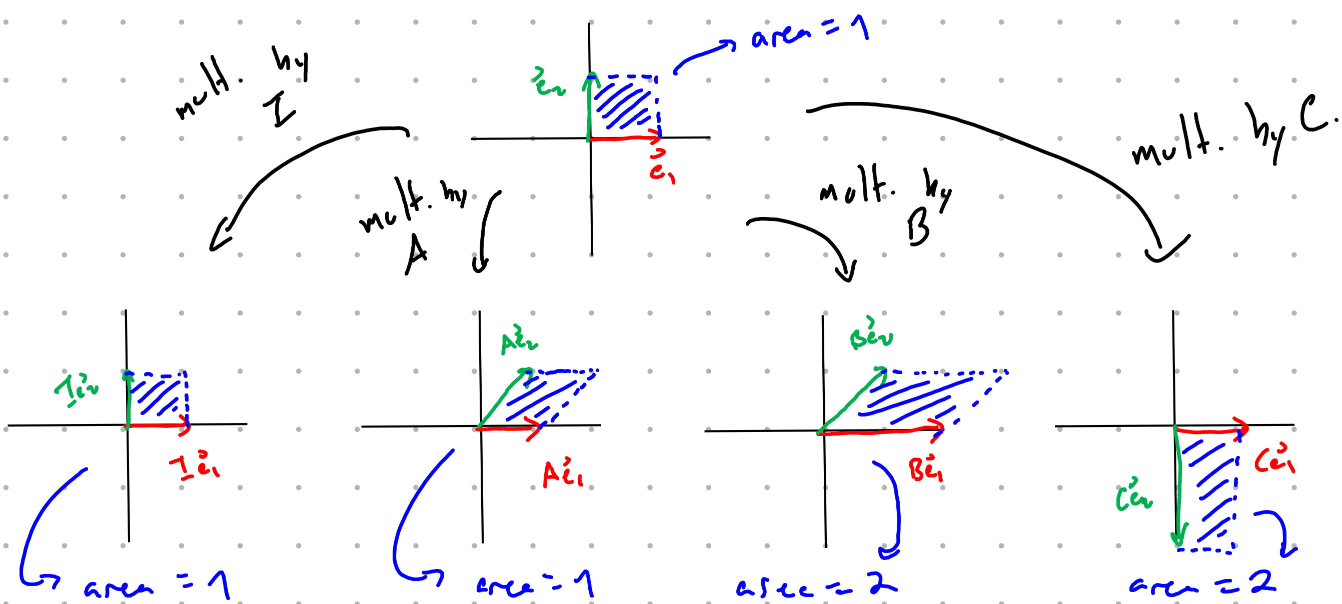
Geometry of the determinant.

We will see that $\det(A)$ informs us about how area changes under the linear transformation defined by A.

Consider the matrices

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

and their action on the square spanned by $\vec{e}_1, \vec{e}_2$ in $\mathbb{R}^2$:



$$\det(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \det(A) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \det(B) = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 \qquad \det(C) = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2$$

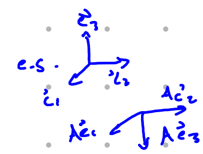
"So we see the determinant tells us the area of the parallelogram spanned by the columns of the matrix (in absolute value!). This is true in general."

Fact: If A is a 2×2 matrix, then $|\det(A)|$ is the area of the parallelogram spanned by the columns of A. If A is 3×3, $|\det(A)|$ is the area of the parallelopiped spanned by the columns of A.

Ex) $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 2 \\ 3 & 1 & -2 \end{bmatrix}$. Note $A\vec{e}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $A\vec{e}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $A\vec{e}_3 = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$ and $\det(A) = -36$.

So the volume of the parallelopiped spanned by the columns of A is $|\det(A)| = -36$.

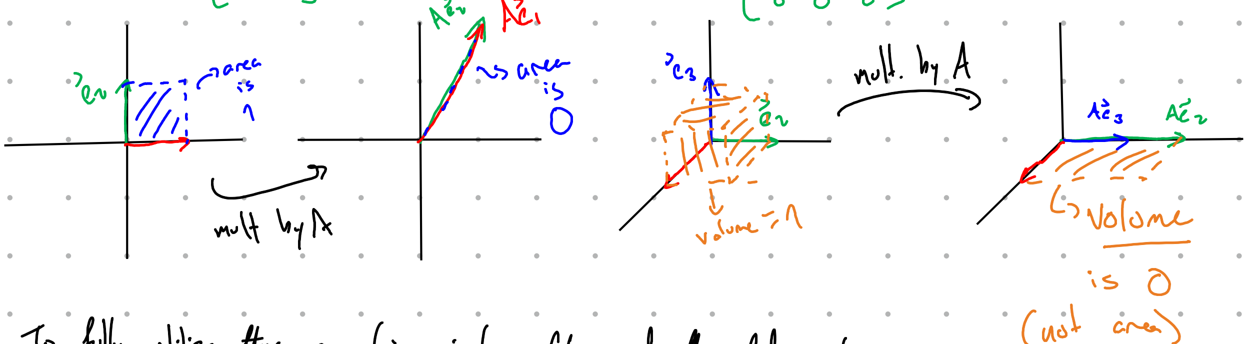
Consider "Volume under a linear transformation" on the course webpage.
We see $\det(A) < 0$ corresponds to changing the orientation of $\vec{e}_1, \vec{e}_2, \vec{e}_3$



Fact: A is invertible if and only if $\det(A) \neq 0$.

Why? $\det(A) = 0$ means the area spanned by the two columns of A is 0
or the volume spanned by the three columns of A is 0.
This indicates A "squishes" 3D down to 1D or 2D for instance, so $T(\vec{x}) = A\vec{x}$ can't be onto, and thus A is not invertible.

Ex) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has $\det(A) = 0$. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $\det(A) = 0$.



To fully utilize this geometric interpretation of the determinant we generalize this relationship between the area of the "unit square" and any parallelogram in the plane.

Fact: If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is determined by a 2×2 matrix A for every parallelogram in $\mathbb{R}^2$ S, we have

$$\text{"area of } T(S)\text{"} = |\det A| \cdot \text{"area of } S\text{"}$$

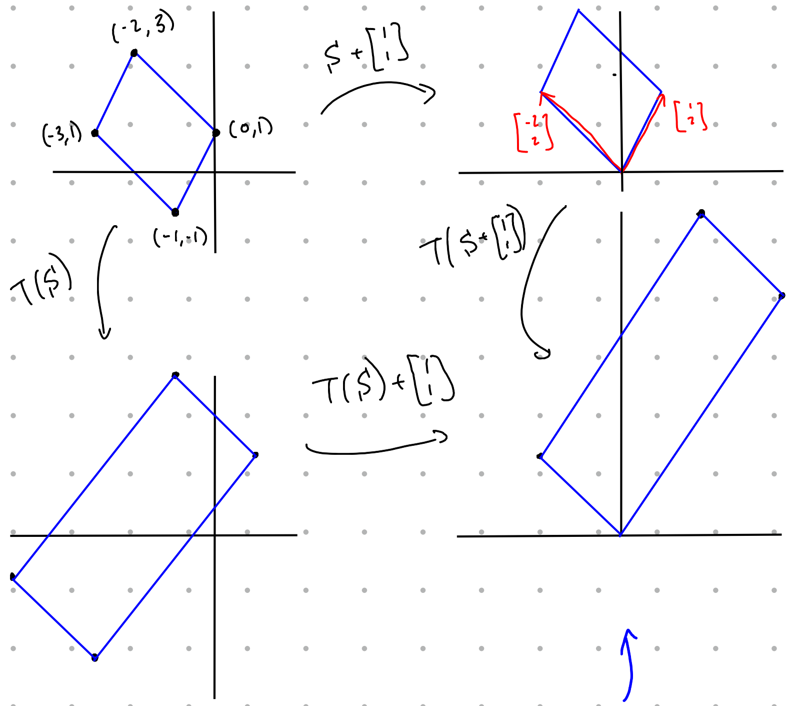
The same is true for $T: \mathbb{R}^3 \to \mathbb{R}^3$, A 3×3, and S a parallelopiped in $\mathbb{R}^3$.
(In fact, we can use this equation to understand 4, 5 and n-dimensional volume!)

Ex Consider the parallelogram S with vertices $(-1,-1)$, $(0,1)$, $(-2,3)$ and $(-3,1)$. Compute the area of S and $T(S)$ if $T:\mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation $T(\vec{x}) = A\vec{x}$ with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Notice if we translate S to a parallelogram with a vertex on the origin, we can compute it's area with the determinant:

$$\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} = 6$$

so area of S is 6.



By the above fact

area of $T(S) = |\det A| \cdot 6$

$= \left|\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\right| \cdot 6$

$= 3 \cdot 6 = \boxed{18}$

notice this is the same as area of $T(S) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$T(S) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is spanned by $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and

$$\begin{vmatrix} 4 & -2 \\ 5 & 2 \end{vmatrix} = 18.$$

"Why do we care about this?

- approximating non-parallelograms by many small parallelograms allow us to estimate any area and how it is changed under a transformation
- this is exactly how we compute integrals by change of variables: the transformation in question is the change of variables and the determinant which measures change in area is the Jacobian" $\rightsquigarrow$ Calc III