Math 2210Q-006/011

Exam II

Nov 15^{th}, 2019

Name: Key

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- If you require extra space for any answer, you may use the back sides of the exam pages. Please indicate when you have done this so that I do not miss any of your work.

ACADEMIC INTEGRITY AGREEMENT

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: ______________________________

(full name)

Questions:	1	2	3	4	5	6	Bonus	**Total**
Points:	15	20	15	20	15	15	10	100
Score:								

Percentage

1. (15 points) (a) (5 points) Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 2 & 6 & 8 \end{bmatrix}$. Calculate the determinant of A.

$$\det A = \begin{vmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 2 & 6 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & -7 & -9 \\ 0 & -2 & -3 \\ 0 & -1 & -1 \end{vmatrix} = - \begin{vmatrix} -2 & -7 & -9 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \end{vmatrix} = - \begin{vmatrix} -2 & -7 & -9 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} -2 & -3 \\ -1 & -1 \end{vmatrix} = -2(-1) = \boxed{2} \qquad = (-1)(-2)(-1)(-1)$$

(b) (5 points) Let $B = \begin{bmatrix} -2 & 0 & -7 & -9 \\ 7 & 3 & 6 & -1 \\ 2 & 0 & 5 & 6 \\ 2 & 0 & 6 & 8 \end{bmatrix}$. Calculate the determinant of B using a cofactor expansion.

$$\det B = \begin{vmatrix} -2 & 0 & -7 & -9 \\ 7 & 3 & 6 & -1 \\ 2 & 0 & 5 & 6 \\ 2 & 0 & 6 & 8 \end{vmatrix} = 3 \begin{vmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 2 & 6 & 8 \end{vmatrix} = 3(\det A) = 3 \cdot 2 = \boxed{6}$$

(c) (2 points) Using the properties of the determinant, calculate $\det 2A$.

$$\det 2A = 2^3 \cdot \det A = 2^3 \cdot 2 = \boxed{16}$$

(d) (2 points) Using the properties of the determinant, calculate $\det B^3$.

$$\det B^3 = (\det B)^3 = 6^3 = \boxed{216}$$

(e) (1 point) Is B invertible? Justify your answer.

Yes. $\det B = 6 \neq 0$.

2. (20 points) Consider $\mathbb{P}_2$ the vector space of all polynomials of degree at most 2.

(a) (5 points) Show that the set of polynomials $\{1+t^2, t-3t^2, 1+t-3t^2\}$ is linearly independent.

$S = \{1, t, t^2\}$ is the standard basis of $\mathbb{P}_2$.

$$\left[[1+t^2]_S \; [t-3t^2]_S \; [1+t-3t^2]_S \right]$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow$$ has 3 pivots, by I.M.T. has linearly independent columns.

These polynomials have linearly independent coordinate vectors and thus must be linearly independent.

(b) (5 points) Let U be the collection of all polynomials of the form $p(t) = a + t^2$ for any a in $\mathbb{R}$. Is U a subspace of $\mathbb{P}_2$? If so, show U shows the three properties of a subspace. If not, show one of these properties fail.

No. $p(t) = a + t^2$ and $q(t) = b + t^2$ are in U

but $(p+q)(t) = (a+b) + 2t^2$ is not in U.

(c) (5 points) Define a transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(1) \\ p(0) \end{bmatrix}$. Show that T is linear. (That is, for any polynomials p, q and scalars c we have $T(p+q) = T(p) + T(q)$ and $T(cp) = cT(p)$.)

Let $p(t) = a_0 + a_1 t + a_2 t^2$, $q(t) = b_0 + b_1 t + b_2 t^2$

So $(p+q)(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$ and

$(cp)(t) = ca_0 + ca_1 t + ca_2 t^2$

Note $p(0) = a_0$, $p(1) = a_0 + a_1 + a_2$, $q(0) = b_0$, $q(1) = b_0 + b_1 + b_2$

$(p+q)(0) = a_0 + b_0$, $(p+q)(1) = a_0 + b_0 + a_1 + b_1 + a_2 + b_2$

and $(cp)(0) = ca_0$, $(cp)(1) = ca_0 + ca_1 + ca_2$.

Thus: $T(p+q) = \begin{bmatrix} a_0 + b_0 + a_1 + b_1 + a_2 + b_2 \\ a_0 + b_0 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + a_2 \\ a_0 \end{bmatrix} + \begin{bmatrix} b_0 + b_1 + b_2 \\ b_0 \end{bmatrix} = T(p) + T(q)$

$T(cp) = \begin{bmatrix} ca_0 + ca_1 + ca_2 \\ ca_0 \end{bmatrix} = c\begin{bmatrix} a_0 + a_1 + a_2 \\ a_0 \end{bmatrix} = cT(p)$. ✓

(d) (5 points) Give an explicit description of the range and kernel of T via spanning sets. (*Hint:* Both require at least two linearly independent vectors.)

If $p(t) = a + bt + ct^2$ then $T(p) = \begin{bmatrix} p(1) \\ p(0) \end{bmatrix} = \begin{bmatrix} a + b + c \\ a \end{bmatrix}$.

So an element of the range is $a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

for some a, b, c: $\boxed{\text{Range}(T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \mathbb{R}^2.}$

An element of the Kernel is a polynomial s.t.

$T(p) = \begin{bmatrix} a + b + c \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ b + c = 0 \Rightarrow -b = c. \end{matrix}$

So $p(t)$ is in the Kernel if it is of the form

of $bt + (-b)t^2 = bt - bt^2 = b(t - t^2)$.

So $\boxed{\text{Kernel}(T) = \text{Span}\{t - t^2\}}$

3. (15 points) The following matrices are row equivalent:

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) For what k is Col(A) a subspace of $\mathbb{R}^k$? For what k is Nul(A) a subspace of $\mathbb{R}^k$?

A is 3×4 so

Col(A) in $\mathbb{R}^3$, Nul(A) in $\mathbb{R}^4$

(b) (5 points) Find a basis for Col(A).

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}.$$

Use the pivot columns of A

(c) (5 points) Find a basis for Nul(A). $A\vec{x} = \vec{0}$ if

$$\vec{x} = \begin{bmatrix} -9x_3 \\ -5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ -5 \\ 1 \\ 0 \end{bmatrix}.$$

So $\left\{ \begin{bmatrix} -9 \\ -5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis.

4. (20 points) (a) (5 points) Let $A = \begin{bmatrix} 5 & 2 \\ 7 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Solve the equation $A\mathbf{x} = \mathbf{b}$ using Cramer's rule. No credit will be given for other methods of solution.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ where } x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix}} = \frac{1}{-9}$$

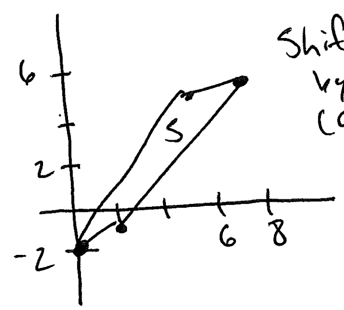
$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{\begin{vmatrix} 5 & 3 \\ 7 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix}} = \frac{-16}{-9}$$

(b) (5 points) Observe that $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. Give the coordinate vector for $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ from above relative to $\mathcal{B}$.

From part (a); $A = P_{\mathcal{B}}$ so

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1/9 \\ 16/9 \end{bmatrix}.$$

(c) (5 points) Compute the area of the parallelogram S with vertices $(0,-2), (5,5), (2,-1), (7,6)$.



Shift up by $(0,2)$ $\quad (0,0), (5,7), (2,1), (7,8)$

$$\text{Area of } S = \left| \det \begin{bmatrix} 5 & 2 \\ 7 & 1 \end{bmatrix} \right| = |-9| = \boxed{9}$$

(d) (5 points) Suppose $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ is another basis of $\mathbb{R}^2$ with change of coordinate matrix $P_{\mathcal{C}}$. Is $P_{\mathcal{C}}$ invertible? Justify your answer.

Yes. $P_{\mathcal{C}} = [\vec{c}_1 \ \vec{c}_2]$ has linearly independent columns because $\mathcal{C}$ is a basis.

By the I.M.T., $P_{\mathcal{B}}$ must be invertible.

5. (15 points) Let $D_{2\times 2}$ be the collection of all 2×2 diagonal matrices. That is

$$D_{2\times 2} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}.$$

With addition and scalar multiplication defined in the usual way for 2×2 matrices, show that $D_{2\times 2}$ verifies axioms 2 and 4 of the vector space axioms and is a subspace of $M_{2\times 2}$.

(a) (5 points) Axiom 2: for all vectors $\mathbf{u}, \mathbf{v}$, we have $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.

Let $\vec{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\vec{v} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ be arbitrary elements of $D_{2\times 2}$.

$$\vec{u}+\vec{v} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} = \begin{bmatrix} c+a & 0 \\ 0 & d+b \end{bmatrix}$$

$$= \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \vec{v} + \vec{u}.$$

✓

(b) (5 points) Axiom 4: there is a zero vector $\mathbf{0}$ in V such that for any vector $\mathbf{u}$, we have $\mathbf{u}+\mathbf{0} = \mathbf{u}$.

With $\vec{u}$ as above, if $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then

$$\vec{u}+\vec{0} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+0 & 0 \\ 0 & b+0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \vec{u}$$

as desired. ✓

(c) (5 points) Verify that $D_{2\times 2}$ is a subspace of $M_{2\times 2}$.

Note as $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ we have

$$D_{2\times 2} = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

6. (15 points) Indicate whether each statement is true or false by circling **True** or **False** appropriately.

(a) (3 points) If A and B are $n \times n$ matrices then $\det(AB) = \det(BA)$.

True **False**

(b) (3 points) If f is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and $f(t) = 0$ for some t, then f is the zero vector in V.

True **False**

(c) (3 points) The kernel of a linear transformation is not a vector space.

True **False**

(d) (3 points) If $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

True **False**

(e) (3 points) Suppose $\mathcal{B}$ is a basis for $\mathbb{R}^n$ and let $P_{\mathcal{B}}$ be the change of coordinate matrix for $\mathcal{B}$. Then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

True **False**

7. (**Bonus:** 5 points) Assume A is an $n \times n$ invertible matrix. Prove that $\det(A^{-1}) = (\det(A))^{-1}$.

(Note: By $(\det(A))^{-1}$, we mean $\frac{1}{\det(A)}$.)

Notice as I_n is diagonal, $\det(I_n) = 1$.

So $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$

which implies $\frac{1}{\det(A)} = \det(A^{-1})$.

8. (**Bonus:** 5 points) Recall that a scalar λ is an *eigenvalue* of an $n \times n$ matrix A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

has a nontrivial solution. Suppose that a specific matrix A has $\lambda = 0$ as an eigenvalue. Is A invertible or not? Justify your answer.

No. If $\lambda = 0$ is an eigenvalue of A then

$$A\vec{x} = 0\vec{x} = \vec{0}$$

has a nontrivial solution. By the I.M.T. A must not be invertible.