Math 2210Q-006/011 **Exam I** **Oct 11^{th}, 2019**

Name: Key

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.

ACADEMIC INTEGRITY AGREEMENT

I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed: ______________________________
(full name)

Questions:	1	2	3	4	5	6	Bonus	**Total**
Points:	20	15	15	10	15	10	10	85
Score:								

Percentage

1. (20 points) Consider the following linear system of equations

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 3 \\ 4x_1 - x_2 + 6x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 5 \end{aligned}$$

(a) (5 points) Write a matrix equation and a vector equation which are equivalent to this system.

$$\begin{bmatrix} 1 & -1 & 3 \\ 4 & -1 & 6 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix}$$

(b) (5 points) Solve the system of equations and give your solution in parametric vector form.

$$\begin{bmatrix} 1 & -1 & 3 & 3 \\ 4 & -1 & 6 & 9 \\ 2 & -1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 3 & -6 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 &= 2 - x_3 \\ x_2 &= -1 + 2x_3 \\ x_3 &\text{ free} \end{aligned} \qquad \text{so} \quad \boxed{\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}.$$

(c) (5 points) Give the solution of the associated homogeneous equation to this system.

$$\boxed{\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}$$

(d) (5 points) Are the vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$ linearly independent? If so, justify your answer. If not, give a linear dependence relation among these vectors.

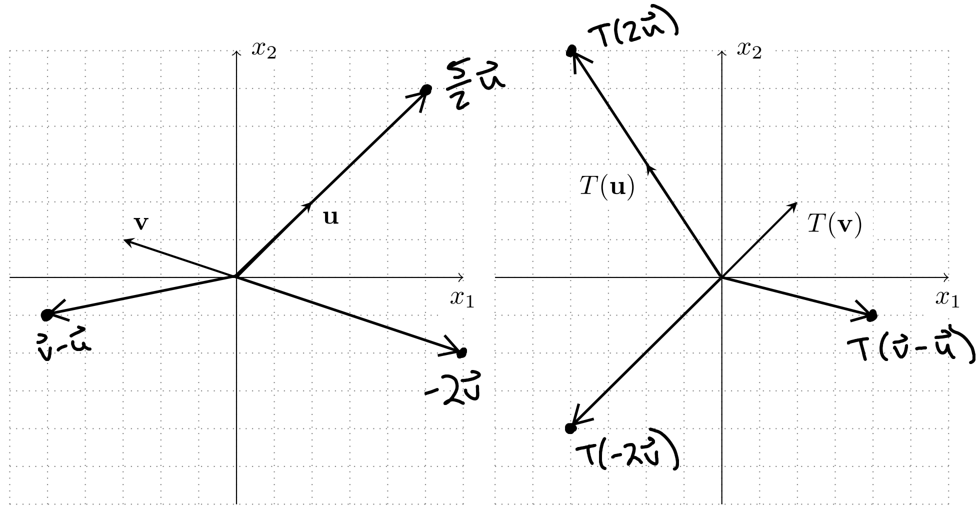
No if $A = [\vec{a}_1 \; \vec{a}_2 \; \vec{a}_3]$ then $A\vec{x} = \vec{0}$

so long as $\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. $x_3 = 1 \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

so

$$\boxed{-\vec{a}_1 + 2\vec{a}_2 + \vec{a}_3 = \vec{0}, \text{ i.e. } -\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

2. (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Consider the vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$ given below and to the right. Their images $T(\mathbf{u})$ and $T(\mathbf{v})$ are given below and to the left.



(a) (5 points) In the left plot, carefully sketch **and label** the vectors $\frac{5}{2}\mathbf{u}$, $-2\mathbf{v}$ and $\mathbf{v} - \mathbf{u}$.

(b) (5 points) In the right plot, carefully sketch **and label** the vectors $T(2\mathbf{u})$, $T(-2\mathbf{v})$ and $T(\mathbf{v} - \mathbf{u})$.

(c) (5 points) Suppose $\{\mathbf{a}_1, \mathbf{a}_2\}$ are a linearly dependent set of vectors from $\mathbb{R}^2$. Is $\{T(\mathbf{a}_1), T(\mathbf{a}_2)\}$ linearly dependent? Justify your answer.

Yes, if $c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{0}$ is a linear dependence relation for $\{\vec{a}_1, \vec{a}_2\}$ then $c_1T(\vec{a}_1) + c_2T(\vec{a}_2) = \vec{0}$ is a linear dependence relation for $\{T(\vec{a}_1), T(\vec{a}_2)\}$

as

$$c_1T(\vec{a}_1) + c_2T(\vec{a}_2) \stackrel{\uparrow}{=} T(c_1\vec{a}_1 + c_2\vec{a}_2) = T(\vec{0}) \stackrel{\uparrow}{=} \vec{0}$$

because T is linear.

3. (15 points) Consider the matrices $A = \begin{bmatrix} 2 & 2 \\ 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$.

(a) (5 points) Circle each expression that is defined.

(i) $A + B$ (ii) $B - C$ (iii) $C + B^T$ (iv) $CA - A$

(v) AB (vi) BAC (vii) CA^T (viii) $(BC)^T$

(b) (5 points) Compute $C(B + C^T)$.

$$B + C^T = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 2 & 0 \end{bmatrix}$$

$$C(B + C^T) = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 7 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 2 & 0 & 7 \\ 4 & -2 & 14 \\ 6 & 2 & 21 \end{bmatrix}}$$

(c) (5 points) Find A^{-1} and solve the three matrix equations

(i) $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (ii) $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (iii) $A\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 2 \\ 4 & 6 \end{bmatrix} \Rightarrow |A| = 4 \Rightarrow A^{-1} = \frac{1}{4} \begin{bmatrix} 6 & -2 \\ -4 & 2 \end{bmatrix}$$

i) $\vec{x} = A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1 \end{bmatrix}$

ii) $\vec{x} = A^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

iii) $\vec{x} = A^{-1} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{4} \left(-2 \begin{bmatrix} 6 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -9/2 \\ 7/2 \end{bmatrix}$

4. (10 points) (a) (5 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix}.$$

Is T one-to-one? Is T onto? Justify your answers.

$A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$ is the standard matrix of T.

One-to-one? No. 3^{rd} column has no pivot.
(so the columns are linearly dependent)

onto? Yes. Each row has a pivot.
(so the columns span $\mathbb{R}^2$).

(b) (5 points) Show that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{bmatrix}$$

is **not** linear.

If T is linear then $T(\vec{0}) = \vec{0}$, but here

$T(\vec{0}) = T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \neq \vec{0}$. So T cannot be linear.

(One could also show, for example, that here

$T(\vec{e}_1 + \vec{e}_2) \neq T(\vec{e}_1) + T(\vec{e}_2)$.)

5. (15 points) Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (5 points) Assume A is invertible and find the third column of A^{-1}.

If $A^{-1} = [\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3]$ then $A\vec{x}_3 = \vec{e}_1$

so $[A \ \vec{e}_1] = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \boxed{\vec{x}_3 = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}}.$

(b) (5 points) Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation for which

$$\mathbf{e}_1 \mapsto \mathbf{e}_1 \qquad \mathbf{e}_2 \mapsto \mathbf{e}_3 \qquad \mathbf{e}_3 \mapsto \mathbf{e}_2.$$

Find the standard matrix B of S.

So $S(\vec{e}_1) = \vec{e}_1$ thus $B = [S(\vec{e}_1) \ S(\vec{e}_2) \ S(\vec{e}_3)]$

$S(\vec{e}_2) = \vec{e}_3$,

$S(\vec{e}_3) = \vec{e}_2$

$\boxed{= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$

(c) (5 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with standard matrix A so that $T(\mathbf{x}) = A\mathbf{x}$. Find the standard matrix of the composition of T with S: that is, find a matrix C so that $T(S(\mathbf{x})) = C\mathbf{x}$.

$$T(S(\vec{x})) = T(\underbrace{B\vec{x}}_{\text{because } B \text{ is the standard matrix of } S}) = \underbrace{A \cdot (B\vec{x})}_{\text{because } A \text{ is the standard matrix of } T} = (\underbrace{AB}_{})\vec{x}$$

↳ $C = AB$ is the desired standard matrix.

$$C = AB = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 2 \\ 0 & 1 & 0 \end{bmatrix}}$$

6. (10 points) Indicate whether each statement is true or false by circling **True** or **False** appropriately.

(a) (2 points) If A and B are invertible $n \times n$ matrices, then AB is invertible with $(AB)^{-1} = A^{-1}B^{-1}$.

True **False**

(b) (2 points) If a linear system has free variables then the solution set contains infinitely many solutions.

True **False**

(c) (2 points) The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

True **False**

(d) (2 points) If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

True **False**

(e) (2 points) If one row of an augmentned matrix in echelon form is

$$\begin{bmatrix} 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

then the associated linear system is inconsistent.

True **False**

7. (10 points) **Bonus:** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Justify why $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ if

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

If $\vec{x}$ in $\mathbb{R}^n$, then

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

Thus $T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$

$$= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n)$$

as T is linear

$$= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A\vec{x}.$$

✓