Math 2110-003 Worksheet 16.6 **Name:** ______________________________ **Due:** 11/29/2017

Parametric surfaces and their areas

1. (a) Parametrize the plane that passes through $(0, 0, 6)$ and contains the vectors $\langle 3, 0, -6\rangle$ and $\langle 0, 2, -6\rangle$.

   (b) Find the surface area of the portion of this plane in the first octant.

2. Find a parametric representation of the cylinder $x^2 + y^2 = 4$ where $0 \leqslant z \leqslant 1$.

3. Find two parametrization of the (positive) cone $z = \sqrt{4x^2 + 4y^2}$.
   Hint: You can use both rectangular and cylindrical coordinates.

In the previous question, you could have used the parametrization

$$x = x \qquad y = y \qquad z = f(x, y).$$

Equivalently, $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$. Let's take a closer look at the advantage of this parametrization for finding the surface area of the graph of $f(x, y)$.
Here the parameters are x and y so the surface area of the graph of $f(x, y)$ for (x, y) in some region D is given by $\displaystyle\iint_D |\mathbf{r}_x \times \mathbf{r}_y|\, dA$. Note

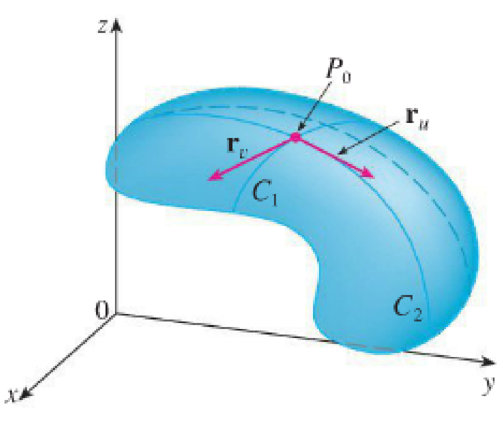
$$\mathbf{r}_x(x, y) = \langle 1, 0, f_x(x, y)\rangle \text{ and } \mathbf{r}_y = \langle 0, 1, f_y(x, y)\rangle$$

and thus $\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1\rangle$. (Check this. This should seem familiar from section 14.4 on tangent planes.) It follows that $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + (f_x)^2 + (f_y)^2}$. So if $z = f(x, y)$ and S is the surface given by the graph of $f(x, y)$ over D then

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

This method of finding surface area is introduced in section 15.5 but we skipped it as it follows so quickly here.

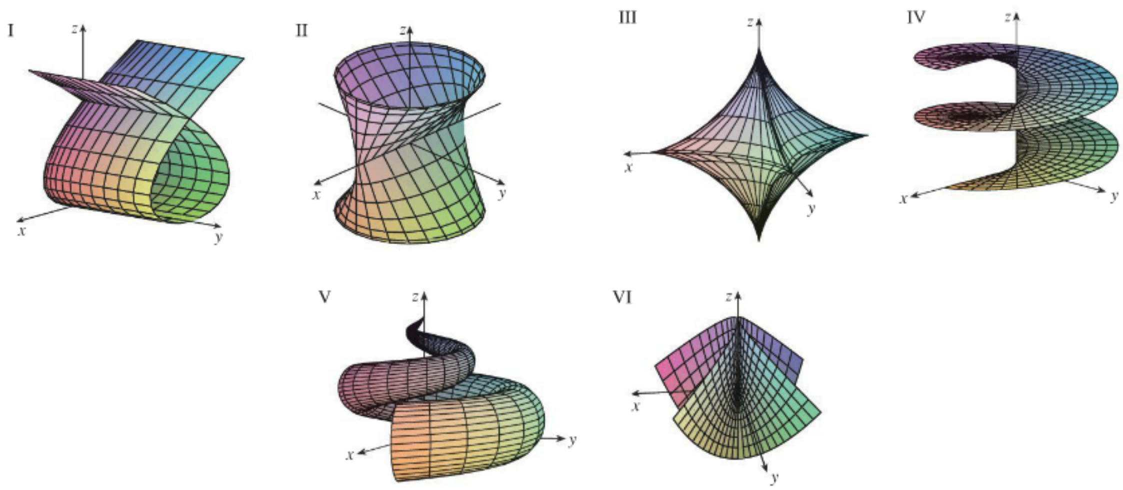
4. In light of the previous paragraph, find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$. (*Hint:* Here $\partial z/\partial x = 2x$.)



As mentioned above, the cross product $r_x \times r_y$ should seem familiar from 14.4. The vector $\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$ is normal to the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ and hence can be used to define the tangent plane there.
For a parametric surface S given by $\mathbf{r}(u, v)$ we can similarly find the tangent plane at any point $\mathbf{r}(u_0, v_0)$. This is because $\mathbf{r}_u(u_0, v_0)$ is the tangent vector of the grid curve $\mathbf{r}(u, v_0)$ at $\mathbf{r}(u_0, v_0)$. Similarly $\mathbf{r}_v(u_0, v_0)$ is the tangent vector of the grid curve $\mathbf{r}(u_0, v)$ at $\mathbf{r}(u_0, v_0)$. See the figure on the left ($\mathbf{r}(u, v_0)$ is C_2 and $\mathbf{r}(u_0, v)$ is C_1). Hence, their cross product $\mathbf{r}_u \times \mathbf{r}_v$, evaluated at (u_0, v_0) is normal to the surface at $\mathbf{r}(u_0, v_0)$. Using this vector, we can find the tangent plane to S at the point $\mathbf{r}(u_0, v_0)$. You essentially did just this on question 5 of the second worksheet on chapter 13.

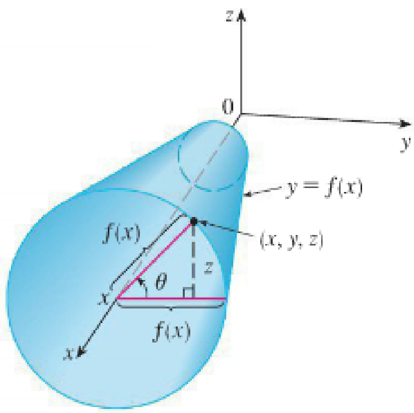
5. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$ and $z = u + 2v$ at the point $(1, 1, 3)$.

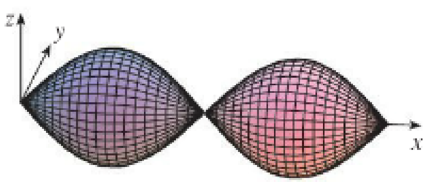
6. Match the following graphs to the parametrization given below.



______ $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$	______ $\mathbf{r}(u,v) = \langle uv^2, u^2v, (u^2 - v^2)\rangle$
______ $\mathbf{r}(u,v) = \langle u^3 - u, v^2, u^2\rangle$	______ $x = (1-u)(3+\cos v)\cos(4\pi u)$ $y = (1-u)(3+\cos v)\sin(4\pi u)$ $z = 3u + (1-u)\sin v$
______ $x = \cos^3 u\cos^3 v$ $y = \sin^3 u\cos^3 v$ $z = \sin^3 v$	______ $x = \sin u$ $y = \cos u\sin v$ $z = \sin v$

The following portion of the worksheet is **optional** but will provide an excellent challenge for your understanding of this, as well as some material from single variable calculus.





Recall surfaces of revolution from single variable calculus. These were generated by taking the graph of a function $y = f(x)$ and either rotating it about the x or y axis. We then studied how to find the volume and surface area of such objects. The volume bounded by the surface obtained by rotating the graph of $f(x)$ from $x = a$ to $x = b$ about the x axis was given by $\int_a^b \pi(f(x))^2\,dx$. (Why is this?) Here we can parametrize this solid of revolution by taking x and θ as parameters. The surface of revolution mentioned above is given by

$$\mathbf{r}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle$$

where $x \in [a, b]$ and $\theta \in [0, 2\pi]$. Explain why this parametrization works based on the figure to the left and above. An example is given in the lower figure: this is the surface obtained by rotating $y = \sin x$ about the x axis.

$$\mathbf{r}(x, \theta) = \langle x, \sin x\cos\theta, \sin x\sin\theta\rangle$$

with $x \in [0, 2\pi]$ and $\theta \in [0, 2\pi]$.

7. In single variable calculus, we had that the surface area of a surface of revolution obtained from rotating the graph of $f(x)$ from $x = a$ to $x = b$ about the x axis was given by

$$A(S) = \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

Above we have a vector function parametrization of this surface. Using the method of computing surface area discussed in this section, prove that this expression in fact computes the surface area. That is, parametrize the appropriate surface of revolution S and find it's surface area. Show it is equal to $\displaystyle\int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$.

8. Consider the function $y = 1/x$ for $x \geqslant 1$. The surface obtained by rotating this graph about the x axis is called Gabriel's horn or Torticelli's trumpet. This object caused a great debate about the philosophical nature of infinity. Here we will investigate the apparent paradoxical nature of this object. A graph of the function and the resulting surface is given below.
   (a) Using the formulas above calculate both the volume and surface area of the initial portion of Gabriel's horn $x \in [1, a)$.
   (b) To find the true volume and surface area of Gabriel's horn, we need take the limit $a \to \infty$ of the corresponding quantities in part (a).
   (c) In part (b), was either quantity finite? Was either infinite? Why might this be considered strange?