Line integrals of scalar functions

1. (a) Evaluate $\int_C 1 - xy\,ds$ if C is the upper half of the circle $x^2 + y^2 = 4$.

   (b) Evaluate $\int_C x \sin z\,ds$ where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, and $z = t$ for $t \in [0, 2\pi)$.

2. (a) Set up the line integral $\int_C 2x + 3y$ if C is the triangle formed by going from $(0,0)$ to $(2,0)$ to $(0,1)$ and back to $(0,0)$.

   (b) Evaluate the line integral $\int_C xy + z$ if C is the curve formed by following the helix in problem 1b from $(1,0,0)$ to $(1,0,2\pi)$ then following the line segment from $(1,0,2\pi)$ back to $(1,0,0)$.

Previously, we have seen that integrals over the constant 1 have a valuable geometric interpretation. Namely, we have seen that

$$\int_I 1\,dx = \text{the length of the interval } I;$$
$$\iint_D 1\,dA = \text{the area of the region } D;$$
$$\iiint_E 1\,dV = \text{the volume of the solid } E.$$

With this in mind, what would you suppose $\int_C 1\,ds$ gives you geometrically? Indeed, it is much the same: if C is parametrized by $\mathbf{r}(t)$ for $t \in [a,b]$ then

$$\int_C 1\,ds = \int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = s = \text{the arc length of } C$$

3. (a) Compute $\int_C ds$ where C is the circle of radius 5 centered at $(1,1)$.

   (b) Compute $\int_C \frac{e}{8\pi}\,ds$ where C is the square formed by going from $(0,0)$ to $(2,0)$ to $(2,2)$ to $(0,2)$ and back to $(0,0)$.

Recall that we define line integrals with respect to a single variable by

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt \qquad \text{and} \qquad \int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt$$

and for functions $P(x,y)$ and $Q(x,y)$, we abbreviate a sum of these integrals by

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy.$$

4. (a) Let C be the line segment from $(1,1)$ to $(4,0)$. Evaluate $\int_C x^2 dy$.

   (b) Let C be the graph of $y = x$ from $(2,2)$ to $(4,4)$. Evaluate $\int xy\, dx + \cos x\, dy$

We can define line integrals over space curves with respect to a single variable similarly:

$$\int_C f(x,y,z)\, dx = \int_a^b f(x(t), y(t), z(t))x'(t)\, dt;$$

$$\int_C f(x,y,z)\, dy = \int_a^b f(x(t), y(t), z(t))y'(t)\, dt;$$

$$\int_C f(x,y,z)\, dz = \int_a^b f(x(t), y(t), z(t))z'(t)\, dt.$$

Likewise, we abbreviate sums of these integrals similarly. If P, Q and R are functions of x, y and z then we write

$$\int_C P(x,y,z)\, dx + \int_C Q(x,y,z)\, dy + \int_C R(x,y,z)\, dz = \int_C P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz.$$

5. Evaluate $\int_C y\, dx + z\, dy + x\, dz$, where C is the line segment from $(2,0,0)$ to $(3,4,5)$.

   Hint: You may wish to compare your work with example 6 from section 16.2 of your textbook.