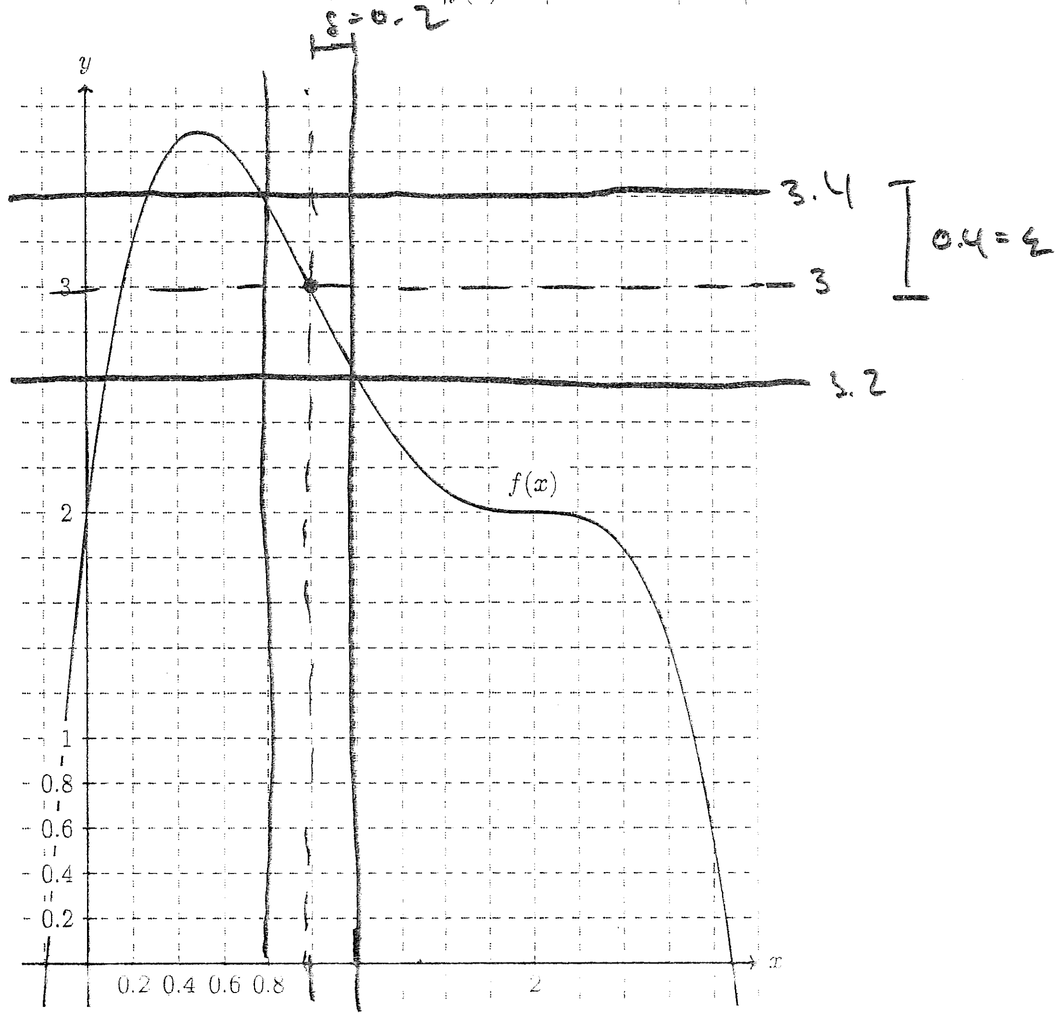
Calc β, BRIDGE 2018 Name: Key Due: Monday, July 9

Homework #3: The precise definition of the limit

Note: Your work can only be assessed if it is legible. You must show all of you work on all problems save 1 and 5. You do not need a calculator to complete this assignment.

1. Consider the function $f(x)$ whose graph is given below. It is clear that $\lim_{x\to 1} f(x) = 3$. Hence, given any $\varepsilon > 0$, we should be able to find a $\delta > 0$ such that $|f(x) - 3| < \varepsilon$ when $|x - 1| < \delta$.

[redacted]



Let $\varepsilon = 0.4$. Find an appropriate δ such that $|x - 2| < \delta$ implies $|f(x) - 3| < \varepsilon = 0.4$.

So if $\boxed{\delta = 0.2}$, then $f(x)$ is within $\varepsilon = 0.4$ of 3.

2. Let $f(x) = 2x + 3$ and $\varepsilon = 0.5$. We consider $\lim_{x \to 0} 2x + 3$.

[redacted] (a) Find $\lim_{x \to 0} 2x + 3$.

[redacted] $\lim_{x \to 0} 2x + 3 = 2(0) + 3 = \boxed{3}$

[redacted] (b) Let $L = \lim_{x \to 0} 2x + 3$, the number you found in the previous part. Find a number $\delta > 0$ such that if $|x - 0| < \delta$ then $|f(x) - L| < \varepsilon$.

$\varepsilon = 0.5$, $L = 3$, need δ s.t. $|x| < \delta$ implies $|f(x) - 3| < 0.5$

$|f(x) - 3| = |2x + 3 - 3| = |2x| < 0.5$ if $|x| < 0.25$.

So, choose $\boxed{\delta = 0.25}$

3. Let $f(x) = x^2 - 2x + 6$ and note $\lim_{x \to 1} x^2 - 2x + 6 = 5$. Here we will work through verifying this fact. To do this, we will need to find for *any* $\varepsilon > 0$ a related $\delta > 0$ such that

$$\text{if } |x - 1| < \delta \text{ then } |f(x) - 5| < \varepsilon.$$

[redacted] (a) We begin by studying a few concrete examples. Let $\varepsilon = \frac{1}{4}$. Find an appropriate value for δ.

$\varepsilon = \frac{1}{4}$, need δ s.t. $|x - 1| < \delta \implies |x^2 - 2x + 6 - 5| < \frac{1}{4}$.

$|x^2 - 2x + 6 - 5| = |(x-1)^2| = |x - 1|^2 < \frac{1}{4}$ if $|x - 1| < \frac{1}{2}$.

choose $\delta = \frac{1}{2}$.

[redacted] (b) Repeat for $\varepsilon = \frac{1}{25}$.

$\boxed{\text{Similarly,}}$ $\delta = \frac{1}{5}$.

use your discretion if they showed enough work.

[redacted] (c) Repeat for $\varepsilon = \frac{1}{100}$.

$\boxed{\text{Similarly,}}$ $\delta = \frac{1}{10}$.

[redacted] (d) Is there a common relationship between each ε and δ you have found?

Yes. δ seems to be the square-root of ε.

[redacted] (e) Let $\varepsilon > 0$ be arbitrary. Find a $\delta > 0$ (in terms of ε) that guarantees $|f(x) - 5| < \varepsilon$ when $|x - 1| < \delta$.

Need $\delta > 0$ s.t. $|f(x) - 5| < \varepsilon$ when $|x - 1| < \delta$.

Need $|f(x) - 5| = |x^2 - 2x + 6 - 5| = |x - 1|^2 < \varepsilon$

if $\underline{|x - 1| < \sqrt{\varepsilon}}$.

So choose $\boxed{\delta = \sqrt{\varepsilon}.}$

[redacted] (f) Bonus:
Use the previous part to *prove* $\lim_{x \to 1} x^2 - 2x + 6 = 5$.

(To get started, write something like "*Let ε be any positive number. Then choose δ to be ....*")

Proof. Let $\varepsilon > 0$. Set $\delta = \sqrt{\varepsilon}$.

We claim if $|x - 1| < \delta = \sqrt{\varepsilon}$, then $|f(x) - 5| < \varepsilon$.

To see this, note

$$|f(x) - 5| = \cdots = |x - 1|^2 \underset{\substack{\uparrow \\ \text{choice} \\ \text{of} \\ \delta}}{<} \delta^2 = \sqrt{\varepsilon}^2 = \sqrt{\varepsilon}.$$

This verifies $\lim_{x \to 1} f(x) = 5$. ✓

[redacted]

4. **Bonus:** Using the precise definition of the limit, prove that $\lim_{x \to -3} 1 - 4x = 13$.

Before starting the proof, we figure what δ should be.

Need $|x+3| < \delta \Rightarrow |1 - 4x - 13| < \varepsilon$.

So

$$|1-4x-13| = |-12-4x| = |-4(x+3)| = 4|x+3| < \varepsilon.$$

So, need $|x+3| < \frac{\varepsilon}{4}$ to make this work.

Let's pick $\delta = \frac{\varepsilon}{4}$.

Proof. Let ε be any pos. number.

Choose $\delta = \frac{\varepsilon}{4}$.

We claim if $|x+3| < \delta = \frac{\varepsilon}{4}$ then $|1-4x-13| < \varepsilon$.

To see this, note

$$|1-4x-13| = |-4(x+3)| = 4|x+3| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

This verifies that $\lim_{x \to -3} 1 - 4x = 13$ by the definition of the limit.