Homework #17 - Key - Integration via substitution

1) (a) $\int (2x+1)e^{x^2+x+7}\,dx = \int e^u\,du$

$u = x^2+x+7$

$du = 2x+1\,dx$

$= e^u + c = e^{x^2+x+7} + c$

(b) $\int \frac{x^3}{1-x^4}\,dx = \int \frac{\frac{-1}{4}du}{u} = \frac{-1}{4}\int \frac{1}{u}\,du = \frac{-1}{4}\ln u + c$

$= \frac{-1}{4}\ln(1-x^4) + c.$

$u = 1-x^4$

$du = -4x^3\,dx$

$\Rightarrow \frac{-1}{4}du = x^3\,dx$

(c) $\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du = -\cos u + c$

$= -\cos(\ln x) + c.$

$u = \ln x$

$du = \frac{1}{x}dx$

(d) $\int \frac{3}{x\ln x}\,dx = \int \frac{3}{u}\,du = 3\ln u + c$

$= 3\ln(\ln x) + c.$

$u = \ln x$

$du = \frac{1}{x}dx$

(e) $\int e^{-x}\,dx = \int -e^u\,du = -e^u + c = -e^{-x} + c.$

$u = -x$

$du = -dx$

so $-du = dx$

(f) $\int \frac{\cos x}{e^{\sin x}}\,dx = \int \frac{1}{e^u}\,du = \int e^{-u}\,du \overset{\substack{\text{prev. prob.}\\ \downarrow}}{=} -e^{-u} + C$

$= -e^{-\sin x} + C.$

$u = \sin x$
$du = \cos x\,dx$

2) (a) $\int_0^1 x^2(1+2x^3)^5\,dx$, $u = 1+2x^3 \Rightarrow du = 6x^2\,dx$ so $\frac{1}{6}du = x^2 dx$

-and- $x=0 \to u = 1+2(0)^3 = 1$

$x=1 \to u = 1+2(1)^3 = 3$

So

$= \int_1^3 \frac{1}{6}(u)^5\,du.$

(b) $\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx$, $u = \cos x \Rightarrow -du = \sin x\,dx$

and, $x=0 \to u = \cos(0) = 1$

$x = \pi/3 \to u = \cos(\pi/3) = \frac{1}{2}$

$= \int_1^{1/2} \frac{-1}{u^2}\,du \quad \left(= \int_{1/2}^1 \frac{du}{u^2} \right).$

(c) $\int_2^3 x e^{-x^2}\,dx$, $u = x^2 \Rightarrow \frac{1}{2}du = x\,dx$

and $x=2 \to u = (2)^2 = 4$
$x=3 \to u = (3)^2 = 9$

$= \int_4^9 \frac{1}{2} e^{-u}\,du.$

3) $\boxed{T}$ If $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\sqrt{x}\,du = dx$

so $2u\,du = dx$.

$x = 0 \Rightarrow u = \sqrt{0} = 0$

$x = 4 \Rightarrow u = \sqrt{4} = 2$ so

$$\int_0^4 f(\sqrt{x})\,dx = \int_0^2 f(u)\,2u\,du.$$

4) (a) $\int x^5\sqrt{1+x^2}\,dx$

$u = 1 + x^2$ so $\frac{1}{2}du = x\,dx$.

$\hookrightarrow x^2 = u - 1$ so $x^4 = (u-1)^2$.

$$= \int \underbrace{x^4}_{(u-1)^2}\,\sqrt{\underbrace{1+x^2}_{u}}\;\underbrace{x\,dx}_{\frac{1}{2}du} = \int \frac{1}{2}(u-1)^2\sqrt{u}\,du$$

$$= \int \frac{1}{2}\left(u^2 - 2u + 1\right)u^{1/2}\,du$$

$$= \int \frac{1}{2}\left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2} - \frac{1}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{2}\left(\frac{2}{7}(1+x^2)^{7/2} - \frac{1}{5}(1+x^2)^{5/2} + \frac{2}{3}(1+x^2)^{3/2} + C\right.$$

(b) $\int \underbrace{(x+3)}_{u+4}\underbrace{(x-1)^5}_{u}\underbrace{dx}_{du}$ $\quad u = x-1 \Rightarrow du = dx$

$\Downarrow$

$u+1 = x$ so $u+1+3 = x+3$

$\Rightarrow u+4 = x+3$

$= \int (u+4)u^5\,du$

$= \int u^6 + 4u^5\,du = \frac{1}{7}u^7 + \frac{2}{3}u^6 + C$

$= \frac{1}{7}(x-1)^7 + \frac{2}{3}(x-1)^6 + C.$

(c) $\int \frac{x}{x+1}\,dx$ $\quad u = x+1 \Rightarrow u-1 = x$ and $du = dx$

$= \int \frac{u-1}{u}\,du = \int 1 - \frac{1}{u}\,du = u - \ln u + C$

$= (x+1) - \ln(x+1) + C.$

(d) $\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$

$u = \sec x + \tan x$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$ $\quad du = \sec x \tan x + \sec^2 x\,dx$

$= \int \frac{1}{u}\,du = \ln u + C$

$= \ln(\sec x + \tan x) + C.$