Calc β, BRIDGE 2018 **Name:** Key **Due:** Tuesday, July 31

Homework #16: The fundamental theorem of calculus

Note: Your work can only be assessed if it is legible.

1. Find the derivative of each of the following functions.

   (a) $\int_0^x \frac{1}{1+t^5}\,dt$

   All are cont. on an appropriate closed interval so FTC applies.

   $$\frac{d}{dx}\int_0^x \frac{1}{1+t^5}\,dt = \frac{1}{1+x^5}$$

   (b) $\int_2^x \sin(e^{2t})\,dt$

   $$\frac{d}{dx}\int_2^x \sin(e^{2t})\,dt = \sin(e^{2x})$$

   (c) $\int_1^{e^x} \ln t\,dt$ if $g(x) = \int_1^x \ln t\,dt$, $g'(x) = \ln x$

   So, $$(g(e^x))' = \frac{d}{dx}\int_1^{e^x} \ln t\,dt = g'(e^x)\cdot(e^x)' = \ln(e^x)\cdot e^x = xe^x.$$

   (d) $\int_0^{x^3} t^2\cos t\,dt$ if $g(x) = \int_0^x t^2\cos t\,dt$, $g'(x) = x^2\cos x$

   so $$(g(x^3))' = \frac{d}{dx}\int_0^{x^3} t^2\cos t\,dt = g'(x^3)\cdot(x^3)' = (x^3)^2\cos(x^3)\cdot 3x^2 = 3x^6\cos(x^3).$$

2. Evaluate the following definite integrals using the fundamental theorem of calculus.

(a) $\int_{-1}^{1} x^{20} + x^{18}\, dx$ $= \dfrac{x^{21}}{21} + \dfrac{x^{19}}{19} \Big|_{-1}^{1}$

$= \dfrac{1}{21} + \dfrac{1}{19} - \left(\dfrac{(-1)^{21}}{21} + \dfrac{(-1)^{19}}{19}\right)$

$= \dfrac{1}{21} + \dfrac{1}{19} + \dfrac{1}{21} + \dfrac{1}{19} = \dfrac{2}{21} + \dfrac{2}{19}.$

(b) $\int_{0}^{1} x^{e} + e^{x}\, dx$ $= \dfrac{x^{e+1}}{e+1} + e^{x} \Big|_{0}^{1}$

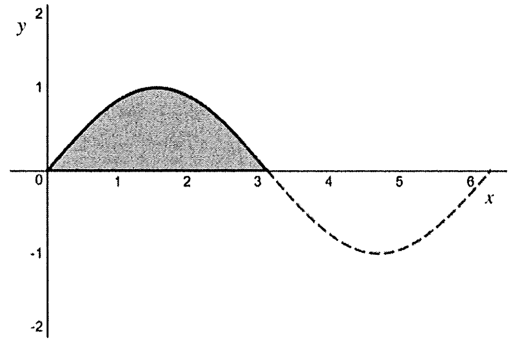
$= \dfrac{1^{e+1}}{e+1} + e - \left(\dfrac{0^{e+1}}{e+1} + e^{0}\right)$

$= \dfrac{1}{e+1} + e - 1.$

(c) $\int_{0}^{1} e^{x+1}\, dx$ $= \int_0^1 e e^x dx = e \int_0^1 e^x dx$

$= e\left(e^x \Big|_0^1\right) = e(e-1).$

3. Find the area bound by "one hump" of $\sin x$. That is, find the area shown below. The plotted graph is $y = \sin x$ on the interval $[0, 2\pi]$.



$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -\cos \pi - (-\cos 0)$$
$$= -(-1) - (-1)$$
$$= 2.$$

4. On the previous homework, you estimated the area under the curve $y = 4 - x^2$ over the interval $[0, 2]$. Use the fundamental theorem of calculus to compute the exact area.

$$\int_0^2 4 - x^2 \, dx = 4x - \frac{1}{3}x^3 \Big|_0^2$$
$$= 8 - \frac{1}{3} \cdot 8 = \frac{24}{3} - \frac{8}{3} = \frac{16}{3}.$$

5. (a) Let $A_0(x) = \int_0^x 1 - t^2\,dt$, $A_1(x) = \int_1^x 1 - t^2\,dt$, and $A_2(x) = \int_2^x 1 - t^2\,dt$.
Compute these explicitly in terms of x using part 2 of the fundamental theorem of calculus.

$$A_0(x) = \int_0^x 1 - t^2\,dt = t - \tfrac{1}{3}t^3 \Big|_0^x = x - \tfrac{1}{3}x^3 = \tfrac{1}{3}x(3 - x^2)$$

$$A_1(x) = t - \tfrac{1}{3}t^3 \Big|_1^x = x - \tfrac{1}{3}x^3 - (1 - \tfrac{1}{3}) = \tfrac{1}{3}x(3 - x^2) - \tfrac{2}{3}$$

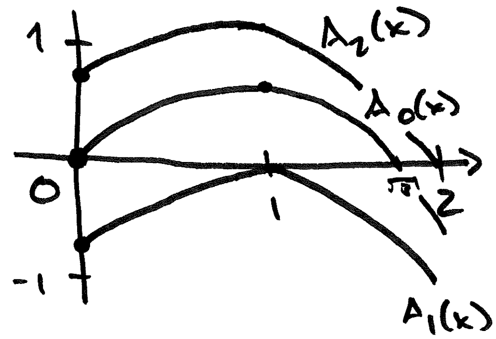
$$A_2(x) = t - \tfrac{1}{3}t^3 \Big|_2^x = x - \tfrac{1}{3}x^3 - (2 - \tfrac{8}{3}) = \tfrac{1}{3}x(3 - x^2) + \tfrac{2}{3}$$

(b) Over the interval $[0, 2]$, use your answers in part (a) to sketch the graphs of $y = A_0(x)$, $y = A_1(x)$, and $y = A_2(x)$ on the same set of axes.

$$A_0(x) = \tfrac{1}{3}x(\sqrt{3} - x)(\sqrt{3} + x)$$

$$A_1(x) = A_0(x) - 2/3$$

$$A_2(x) = A_0(x) + 2/3$$



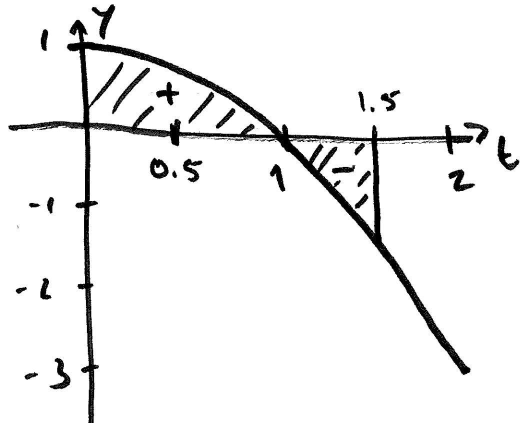
$A_0(1) = \frac{2}{3}$
$A_1(1) = 0$
$A_2(1) = \frac{4}{3}$

(c) How are the three graphs in part (a) related to each other? In particular, what does part 1 of the fundamental theorem of calculus tell you about the graphs in part (a)?

They are the same curve, only differing by vertical shifts.

They all must be antiderivatives of $1 - x^2$, so they must all differ by only a constant.

(d) On a graph of $y = 1 - t^2$, for $0 \leqslant t \leqslant 2$, shade the region with signed area $A_0(1.5)$. Indicate with + and − which area counts positively and which negatively.



6. T/F (with justification) The function $F(x) = \int_0^x \cos(t^2)\,dt$ is an antiderivative of $\cos(x^2)$.

$\boxed{T}$ $\cos(t^2)$ is continuous on $[0,b]$ for any b,

so by FTC (pt 1)

$$F'(x) = \frac{d}{dx}\int_0^x \cos t^2\,dt = \cos x^2.$$

7. T/F (with justification) $\int_{-2}^{2} x^{-4}\,dx = \left.\frac{x^{-3}}{-3}\right|_{-2}^{2} = \frac{-1}{12}$.

$\boxed{F}$ $\frac{1}{x^4}$ is <u>not</u> continuous on $[-2,2]$, ~~since~~ thus

FTC (pt 2) does not apply.

Indeed, since $\lim_{x\to 0} \frac{1}{x^4} = \infty$, this integral

does not exist.

8. Evaluate each indefinite integral.

(a) $\int x^3 + \frac{1}{x^2}\, dx$

$$= \frac{1}{4}x^4 - \frac{1}{x} + C$$

(b) $\int \frac{x^5 - 2\sqrt{x^3}}{x}\, dx$ $= \int \frac{x^5}{x} - 2\,\frac{x^{3/2}}{x}\, dx$

$$= \int x^4 - 2x^{1/2}\, dx = \frac{1}{5}x^5 - 2\left(\frac{2}{3}x^{3/2}\right) + C$$

(c) $\int \frac{\sin x}{\cos^2 x}\, dx$. (*Hint:* $\frac{1}{\cos x} = \sec x$.)

$$= \int \frac{1}{\cos x}\,\frac{\sin x}{\cos x}\, dx = \int \sec x \tan x\, dx$$

$$= \sec x + C.$$

9. Verify by differentiation that the following equation is true:

$$\int \frac{1}{x^2\sqrt{1+x^2}}\, dx = \frac{-\sqrt{1+x^2}}{x} + C.$$

$$\frac{d}{dx}\left(\frac{-\sqrt{1+x^2}}{x} + C\right) = \frac{x\left(-\sqrt{1+x^2}\right)' - (x)'\left(-\sqrt{1+x^2}\right)}{x^2}$$

$$= \frac{1}{x^2}\left(\frac{-x(2x)}{2\sqrt{1+x^2}} + \sqrt{1+x^2}\right) = \frac{1}{x^2}\left(\frac{-x^2+1+x^2}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{x^2\sqrt{1+x^2}}. \checkmark$$

In class, we mentioned that the definite integral of $f'(x)$ over an interval $[a, b]$ represents the *net change* in $f(x)$ due to the fundamental theorem of calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

To see this in a scientific context, note the following implications:

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

  is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If $[C](t)$ is the concentration of the product of a chemical reaction at time t, the rate of reaction is the derivative $d[C]/dt$. So

$$\int_{t_1}^{t_2} \frac{d[C]}{dt}\,dt = [C](t_2) - [C](t_1)$$

  is the change in the concentration of $[C]$ from time t_1 to t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

  is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

  is the net change in population during the time period from t_1 to t_2.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

  is the net change of position, or *displacement,* of the particle during the time period from t_1 to t_2.

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

  is the change in velocity from time t_1 to t_2.

  net

10. Water is released into a tank at the rate $r(t) = 5 + \sqrt{t}$ ft^3/min at time t (in minutes). At $t = 1$ minute, there is 12 ft^3 of water in the tank.

(a) Evaluate $\int_1^9 r(t)\,dt$.

$$= \int_1^9 5 + t^{1/2}\,dt = 5t + \frac{2}{3}t^{3/2}\Big|_1^9$$

$$= 45 + \frac{2}{3}(3)^3 - 5 - \frac{2}{3} = 40 + 18 - \frac{2}{3} = 58 - \frac{2}{3} = 57 + \frac{1}{3}$$

$$\approx 5\cancel{8}7.3$$

(b) In the context given above, what does the value in part (a) tell us?

That from the first minute to the ninth minute, a net change of 57.3 ft^3 of water was added to the tank.

(c) Determine the volume of water in the tank at time $t = 9$ minutes.

$t=1$, $V = 12$ ft^3

so at $t=9$, $V = 12 + \int_1^9 r(t)\,dt \approx 12 + 57.3$

$= 69 + 1/3$. ≈ 69.3