Calc β, BRIDGE 2018 **Name:** Key **Due:** Thursday, July 26

Homework #14: L'Hospital's rule and anti-differentiation

Note: Your work can only be assessed if it is legible.

1. Evaluate the following limits:

(a) $\lim_{t\to 0} \frac{e^{2t}-1}{\sin t}$ $\rightarrow$ indet. form: $\frac{0}{0}$

$$\lim_{t\to 0} \frac{e^{2t}-1}{\sin t} \underset{L'H}{=} \lim_{t\to 0} \frac{2e^{2t}}{\cos t} = \frac{2e^0}{\cos 0} = \boxed{2}$$

(b) $\lim_{x\to 1} \frac{x^2-1}{x^2-x}$

$$\lim_{x\to 1} \frac{x^2-1}{x^2-x} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x(x-1)} = \lim_{x\to 1} \frac{x+1}{x} = \boxed{2}$$

(c) $\lim_{x\to 0} \frac{(x+1)^{13}-13x-1}{x^2}$ $\rightarrow$ $\frac{0}{0}$

$$\lim_{x\to 0} \frac{(x+1)^{13}-13x-1}{x^2} \underset{L'H}{=} \lim_{x\to 0} \frac{13(x+1)^{12}-13}{2x} \underset{L'H}{=} \lim_{x\to 0} \frac{13\cdot 12\cdot (x+1)^{11}}{2}$$

$\frac{0}{0}$

$$= \frac{13\cdot 12\cdot (1)^{11}}{2}$$

$$= 13\cdot 6 = \boxed{78}$$

$\sec 0 = \frac{1}{\cos 0} = 1$

(d) $\lim_{x\to 0} \frac{x - \tan x}{x - \sin x}$ $\to \frac{0}{0}$

$\frac{0}{0}$

$$\lim_{x\to 0} \frac{x - \tan x}{x - \sin x} \underset{L'H}{=} \lim_{x\to 0} \frac{1 - \sec^2 x}{1 - \cos x} \underset{L'H}{=} \lim_{x\to 0} \frac{1 - 2\sec^2 x \tan x}{1 + \sin x}$$

$$= \frac{1 - 2(1)(0)}{1 + 0} = \boxed{1}$$

(e) $\lim_{x\to\infty} x^3 e^{-x^2}$ $= \lim_{x\to\infty} \frac{x^3}{e^{x^2}} \to \frac{\infty}{\infty}$

$\frac{\infty}{\infty}$

$$\underset{L'H}{=} \lim_{x\to\infty} \frac{3x^{\cancel{2}}}{2\cancel{x}e^{x^2}} = \lim_{x\to\infty} \frac{3x}{2e^{x^2}}$$

top $= 3$
bottom $\to \infty$
so

$$\underset{L'H}{=} \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = \boxed{0}$$

(f) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

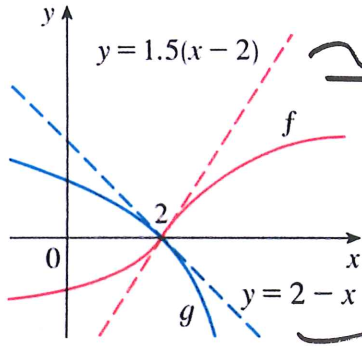
$\frac{0}{0}$

$$\lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$$

$$\underset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} = \frac{1}{1+0} = 1.$$

So $\ln\left(\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x\right) = 1 \Rightarrow \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e^1 = \boxed{e}$

2. Use the graphs of f and g and their tangent lines at $(2, 0)$ to find $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$.



$\Longrightarrow f'(2) = 1.5$

$f(2) = g(2) = 0$

so

$\longrightarrow g'(2) = -1$

$$\lim_{x \to 2} \underbrace{\frac{f(x)}{g(x)}}_{\frac{0}{0}} \underset{\text{L'H}}{=} \lim_{x \to 2} \frac{f'(x)}{g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.5}{-1} = \boxed{-1.5}$$

3. Complete the following table.

Function	Particular antiderivative	Function	Particular antiderivative
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sin x$	$-\cos x$
$\frac{1}{x} \ (x > 0)$	$\ln x$	$\cos x$	$\sin x$
e^x	e^x	$\sec^2 x$	$\tan x$
a^x	$\dfrac{a^x}{\ln a}$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$\dfrac{-1}{\sqrt{1-x^2}}$	$\arccos x$
$\dfrac{1}{1+x^2}$	$\arctan x$		

4. Find the most general antiderivative of the function (use C as any constant).

(a) $f(x) = 1 + 2x + 3x^2 + 7x^3$

$$F(x) = x + x^2 + x^3 + \frac{7}{4}x^4 + C$$

(b) $f(x) = 2\cos x + e^x - 3\sin x$

$$F(x) = 2\sin x + e^x + 3\cos x + C$$

(c) $f(x) = e^2$

$$F(x) = e^2 x + C$$

5. Find a function $f(x)$ such that

$$f''(x) = 8x^3 + 5, \quad f(0) = 1, \quad f'(1) = 8.$$

$f'(x) = \cancel{\tfrac{8}{4}}\,2x^4 + 5x + C$, $f'(1) = 8 = 2(1)^4 + 5(1) + C$

so $C = 1$

$f'(x) = 2x^4 + 5x + 1$

so $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + D$ $\quad f(0) = 1 \Rightarrow D = 1$

so

$$\boxed{f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + 1}$$

6. The fastest recorded speed an F1 car has hit in a race is roughly 231 miles per hour or about 339 ft/s. Suppose you take such a race car onto an unrestricted section of the autobahn[1] to test its brakes. The maximum braking deceleration an F1 car can apply is 6.3 g[2] or about -202 ft/s^2.

   If, at top speed, you apply the brakes constantly at this force to stop the car, how far will you travel before the car comes to a stop?[3]

   Hint: Use feet and seconds in your units.

$a(t) = -202 \qquad v(0) = 339 \qquad s(0) = 0$

So $v(t) = -202t + C \longrightarrow v(t) = -202t + 339$

$s(t) = -101t^2 + 339t.$

Car stops when

$v(t) = -202t + 339 = 0$

so $t = \frac{339}{202}.$

Car has travelled

$s\left(\frac{339}{202}\right) = -101\left(\frac{339}{202}\right)^2 + \frac{339^2}{202}$ ft

$\boxed{\approx 284.46 \text{ ft}}$

7. T/F: (with justification) The antiderivative of $\cos(x^2)$ is $\sin(x^2) + C$.

$\boxed{\text{False}}$

$(\sin(x^2) + C)' = 2x\cos(x^2) \neq \cos(x^2).$

[1] The fastest recorded speed on the autobahn during normal operation (so not for a speed test) is 236 miles per hour.

[2] Here g refers to g-force: 5 g is equivalent to 5 times the acceleration due to gravity on the surface of the earth. For a little context, F1 drivers have reported forced exhalation at 6 g ("having the wind knocked out of you"); 3 g is the maximum acceleration experienced by a Space Shuttle launching or reentering the atmosphere (2.5 - 3 g is what you experience on a carnival "Gravitron"); acrobatic airplane pilots are permitted to force at most 30 g; the Mantis Shrimp (which is a whole footnote on its own) strikes prey with its claw at 10,400 g; and finally 2×10^{11} g is the force of gravity near the surface of a neutron star.

[3] How long would it take you? $\dashrightarrow t = \frac{339}{202} = 1.68$ seconds