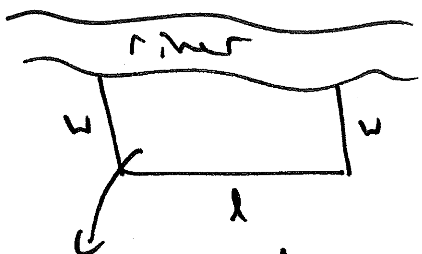
Calc β, BRIDGE 2018 Name: Key Due: Wednesday, July 25

Homework #13: Optimization

Note: Your work can only be assessed if it is legible. You are welcome to use a calculator.

1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



2400 ft of fence, max area.

$l + 2w = 2400.$

so $l = 2400 - 2w$

Rectangular field so

$A = l \cdot w$ ← maximize this while only using 2400 ft of fence.

$\Rightarrow A(w) = (2400 - 2w)w.$ ← find max.

$A = 2400w - 2w^2$

1) crit values.
2) classify each.

so $A'(w) = 2400 - 4w$

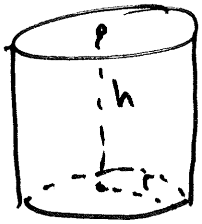
$A'(w) = 0$ if $w = \underbrace{600\text{ ft.}}_{\text{crit. number}}$

But $0 \le w \le 1200$ → end points are crit. numbers.

$A(0) = 0$
$A(600) = 720{,}000$
$A(1200) = 0$

so $\boxed{\begin{array}{l} w = 600 \\ l = 1200 \end{array}}$

2. A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can. ($1000\ cm^3$)



← Volume = 1 L.
Minimize amount of metal:
minimize SA of the can.

Dimensions:
r = radius
h = height

Need $V = (\pi r^2) \cdot h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$

while $S = \underbrace{2(\pi r^2)}_{\text{top \& bottom}} + \underbrace{(2\pi r h)}_{\text{shell}}$ is minimum

$S(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$

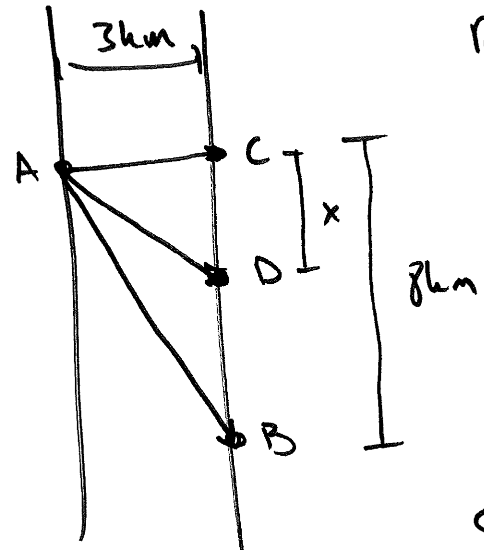
$S'(r) = 4\pi r - \frac{2000}{r^2}$ notice $r > 0$ so crit numbers are $4\pi r - \frac{2000}{r^2} = 0$

$\Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$

$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2}$

$S''(r) = 4\pi + \frac{4000}{r^3} > 0$ for $r > 0$ so CU so min

3. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. (See below.) He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?



Row to C, run to B

0.5 hr   1 hr = 1.5 hr

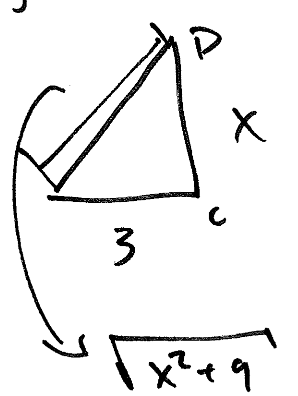
Row to B: $\frac{\sqrt{9+64}}{6} = 1.42$ hr.

Row to D, Run to B.

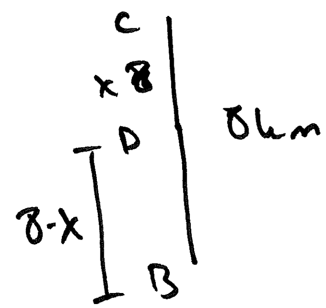
$S = \underbrace{(\sqrt{x^2+9})}_{\text{length to D}}\left(\frac{1}{6}\right) + \underbrace{(8-x)}_{\text{run to B}}\left(\frac{1}{8}\right)$ hrs of travel

where $0 \le x \le 8$

length to D.



run to B



$T'(x) = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$

crit numbers $x=0, x=8$

or

$T'(x) = 0 \Rightarrow \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8} = 0$

if $8x = 6\sqrt{x^2+9}$

$4x = 3\sqrt{x^2+9}$

$\Rightarrow \cancel{64x^2 = 36(x^2+9)}$

$\cancel{64x^2 = 36x^2 +}$

so $16x^2 = 9(x^2+9)$

$16x^2 = 9x^2 + 81$

so $7x^2 = 81 \Rightarrow x = \frac{9}{\sqrt{7}}$ in 0 to 8.

$S(0) = 1.5$ $S(9/\sqrt{7}) = \sqrt{\frac{81}{7}+9}\cdot\frac{1}{6} + \left(8 - \frac{9}{\sqrt{7}}\right)\cdot\frac{1}{8}$

4. Find two numbers whose sum is 100 and whose product is a maximum.

$x + y = 100$ so $P(x) = 100x - x^2$

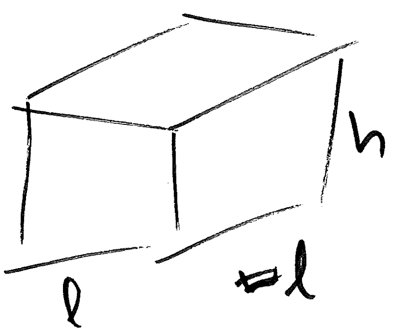
$xy = P$ $P'(x) = 100 - 2x = 0$ if $x = 50$

↑ max

$P''(x) = -2$ so $x = 50$ is a max.

$\Rightarrow \boxed{x = 50, y = 50}$

5. An open box with a square base will be constructed using 300 in^2 of cardboard. What are the dimensions of such a box with the largest possible volume?



$V = l\overset{l}{\cancel{w}}h \leftarrow$ max

$SA = \overset{l^2}{\cancel{lw}} + \underbrace{\cancel{2wh + 2lh}}_{4lh} = 300$

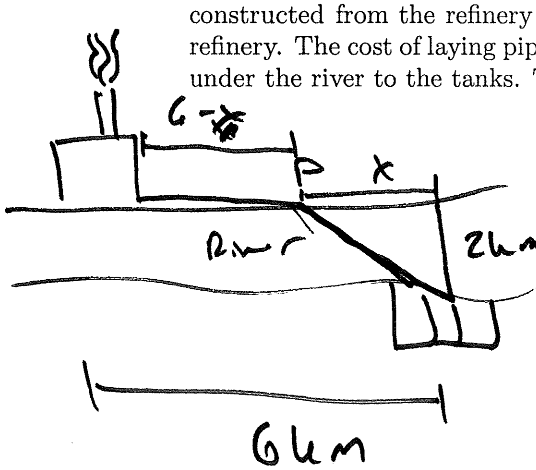
$h = \dfrac{300 - l^2}{4l}$

$\Rightarrow V = \dfrac{300l - l^3}{4} \Rightarrow V' = \frac{1}{4}(300 - 3l^2) = 0$

if $l = w$ so $\boxed{l = 10\text{cm} \Rightarrow h = 5\text{cm}}$

$100 + 40h = 300$

6. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east o the refinery. The cost of laying pipe is $400,000/km over land to a point P on the north bank and $800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?



$$C(x) = (6-x)\,400{,}000 + \sqrt{x^2+4}\,(800{,}000)$$

$$0 \le x \le 6$$

$$C'(x) = \frac{(8\cdot 10^5)\,x}{\sqrt{x^2+4}} - (4\cdot 10^5) = 0 \quad \text{if}$$

$$(8\cdot \cancel{10^5})\,x = 4\cdot\cancel{10^5}\cdot\sqrt{x^2+4}$$

$$2x = \sqrt{x^2+4}$$

$$4x^2 = x^2+4 \quad \Rightarrow \quad x^2 = \frac{4}{3}$$

so $x = \frac{2}{\sqrt{3}}$.

$$C(0) = 4\cdot 10^6$$

$$C\left(\frac{2}{\sqrt{3}}\right) \approx \cancel{3.78564}\ 3.79\cdot 10^6$$

$$C(6) \approx 5.06\cdot 10^6$$

so $\boxed{\frac{2}{\sqrt{3}} \text{ km east}}$.