Calc β, BRIDGE 2018 **Name:** Key **Due:** Tuesday, July 24

Homework # 12: The geometry of a function via its derivatives

Note: Your work can only be assessed if it is legible. A single final answer will be considered a guess and awarded minimal credit.

1. Consider the function $f(x) = 2 + 2x^2 - x^4$.

   (a) Find the critical points of $f(x)$.

$$f'(x) = 4x - 4x^3 = 4x(1-x^2) = 4x(1-x)(1+x) = 0 \text{ if } x = 0, \pm 1$$

so critical points are

$$x=0,\ x=1,\ x=-1.$$

   (b) On which intervals is $f(x)$ increasing? Decreasing?

Interval	$4x$	$1-x$	$1+x$	f'	so on this interval $f(x)$ is
$x<-1$	$-$	$+$	$-$	$+$	increasing
$-1<x<0$	$-$	$+$	$+$	$-$	decreasing
$0<x<1$	$+$	$+$	$+$	$+$	increasing
$1<x$	$+$	$-$	$+$	$-$	decreasing

   (c) On which intervals is $f(x)$ concave up? Concave down?

$$f''(x) = 4 - 12x^2 = 0 \quad \text{if} \quad x = \pm\sqrt{1/3} = \frac{\pm 1}{\sqrt{3}}.$$

So $f''(x) = 12\left(\frac{1}{3} - x^2\right) = 12\left(\frac{1}{\sqrt{3}} - x\right)\left(\frac{1}{\sqrt{3}} + x\right)$

Int.	12	$\left(\frac{1}{\sqrt{3}} - x\right)$	$\left(\frac{1}{\sqrt{3}} + x\right)$	f''	so f is
$x<\frac{-1}{\sqrt{3}}$	$+$	$+$	$-$	$-$	concave down
$\frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$	$+$	$+$	$+$	$+$	concave up
$\frac{1}{\sqrt{3}}<x$	$+$	$-$	$+$	$-$	concave down

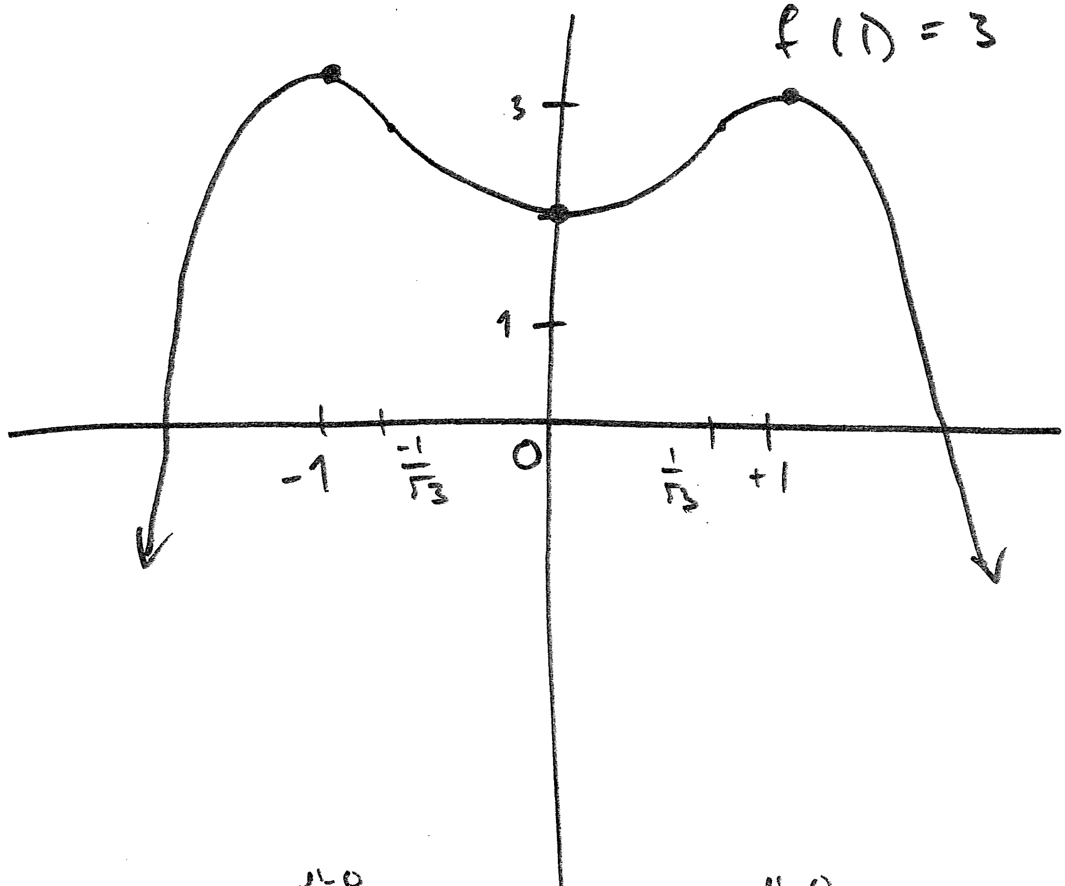
x-values of the

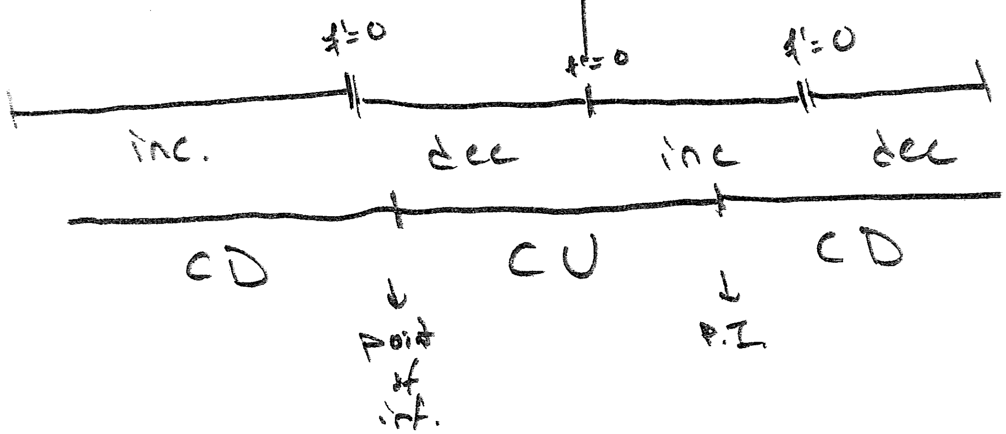
(d) Find the inflection points of $f(x)$.

Based on part (c), f'' changes sign about $x = \frac{-1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ so both are the x-values of inflection points.

(e) Sketch $f(x)$.

$f(-1) = \cdots = 3$
$f(0) = 2$
$f(1) = 3$





the exact x-values of   classify each as a local max or min and

2. For the following functions determine all local maxima and minima. You must verify your claim in each case via a derivative test.

(a) $f(x) = x^5 - 2x^3$

Crit. points: $f'(x) = 5x^4 - 6x^2 = x^2(5x^2-6) \Rightarrow x = 0,\ x = \sqrt{6/5},\ x = -\sqrt{6/5}$

Int:	x^2	$(5x^2-6)$	f'
$x < -\sqrt{6/5}$	+	+	+
$-\sqrt{6/5} < x < 0$	+	−	−
$0 < x < \sqrt{6/5}$	+	−	−
$\sqrt{6/5} < x$	+	+	+

sign change at $x = -\sqrt{6/5}$ ↗ – ↘

no sign change at $x = 0$ ↘ – ↘

sign change at $x = \sqrt{6/5}$ ↘ – ↗

So $f(x)$ has a local max at $x = -\sqrt{6/5}$ and a local min at $x = \sqrt{6/5}$

(b) $f(x) = x - 2\sin x$ for $-2\pi < x < 2\pi$.

Crit points: $f'(x) = 1 - 2\cos(x)$

$f'(x) = 0$ if $0 = 1 - 2\cos x$,

$\cos x = \frac{1}{2}$ if $x = \pi/3,\ 2\pi/3$.

$f''(x) = 2\sin x$. At $x = \pi/3$, $f''(\pi/3) = 2\sin(\pi/3) = \sqrt{3} > 0$ so $f(x)$ is concave up and f has a local min at $x = \pi/3$.

At $x = 2\pi/3$, $f''(2\pi/3) = 2\sin(2\pi/3) = -\sqrt{3} < 0$ so f is concave down and $f(x)$ has a local min at $x = \pi/3$.

(c) $f(x) = e^{-x} - e^{-3x}$ for $x > 0$

Crit. points:

$f'(x) = 3e^{-3x} - e^{-x} = 0$

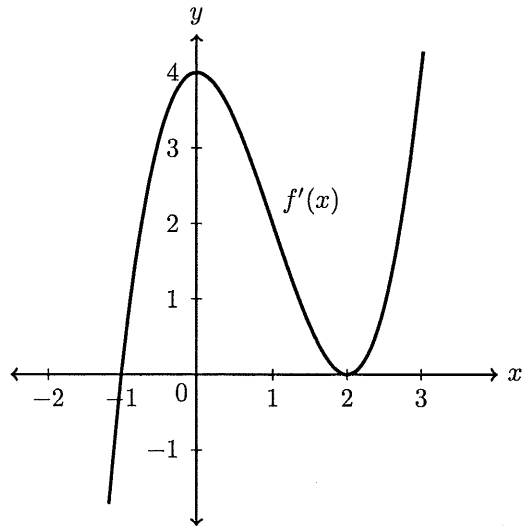
if $3e^{-3x} = e^{-x} \Rightarrow 3 = e^{2x} \Rightarrow x = \frac{\ln 3}{2}$.

$f''(x) = e^{-x} - 9e^{-3x}$ and

$$f''\left(\frac{\ln 3}{2}\right) = e^{-\ln 3/2} - 9e^{-3\ln 3/2} = \frac{1}{\sqrt{3}} - \frac{9}{\sqrt{3^3}} = \frac{1}{\sqrt{3}} - \frac{9}{3\sqrt{3}} = \frac{1-3}{\sqrt{3}} < 0$$

So f is CD at $x = \frac{\ln 3}{2}$ and thus has a local max there.

3. Below is a graph of $y = f'(x)$. Determine the intervals where $f(x)$ is increasing and decreasing, the intervals where $f(x)$ concave up and concave down, the x-values where $f(x)$ has local maxima and minima, and the x-values where $f(x)$ has inflection points.



We see that $f'(x) = 0$ at $x = -1$, $x = 2$.

For $x < -1$, $f'(x) < 0$ so f is decreasing;

$-1 < x < 2$, $f'(x) > 0$ so f is increasing.

$2 < x$, $f'(x) > 0$ so f is increasing.

Since $f'(x)$ changes sign from $-$ to $+$ about $x = -1$, $f(x)$ has a local min at $x = -1$.

Now, $f'(x)$ increases ~~from~~ ~~in~~ on $x < -1$ and $x > 2$

so $f(x)$ is CU there.

$f'(x)$ decreases on $-1 < x < 2$ so $f(x)$ is CD there.

At both $x = -1$ and $x = 2$, $f''(x)$ changes sign $(+ \text{ to } - \text{ to } +)$ so both values are the locations of inflection points.

4. (a) T/F (with justification) If $f(x)$ is a differentiable function on (a, b) and $f'(c) = 0$ for some c in (a, b) then $f(x)$ has a local maximum or minimum value at $x = c$.

False. If $f(x) = x^3$, $f'(x) = 3x^2$ and

$f'(0) = 0$ but f has no max or min at $x = 0$.

(b) T/F (with justification) If a functin $f(x)$ on the interval $(-1, 1)$ is twice differentiable and $f''(c) = 0$ for some c in $(-1, 1)$ then $f(x)$ has an inflection point at $x = c$.

False. If $f(x) = x^4$, then

$f'(x) = 4x^3$ and $f''(x) = 12x^2$.

Note $f''(x) = 0$ but if

	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	+	0	+
f	CU		CU

Since f'' doesn't change sign at $x = 0$,
$f(x)$ does not have an inflection point.