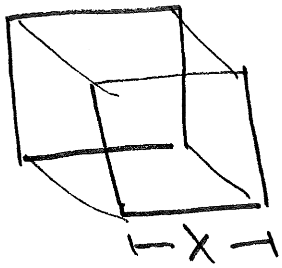
Calc β, BRIDGE 2018 **Name:** Key **Due:** Monday, July 22

Homework #10: Linearization and related rates

Note: This portion will be done in class.

1. Consider a cube of side length x which expands over time. Give the rate at which the volume of the cube V increases in terms of the rate at which x changes.



V = volume

x = side length

$V = x^3$.

So if $\frac{dx}{dt}$ = rate side length changes

then as $V = x^3$

$$\frac{d}{dt}(V) = \frac{d}{dt}(x^3)$$

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}$$

so $V = x^3$.

2. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast it the radius of the balloon increasing when the diameter is 50 cm?

r = radius at time t

V = volume at time t

know! $\frac{dV}{dt} = 100 \ cm^3/s$

want $\frac{dr}{dt}$ when diam is 50cm

so $V = \frac{4}{3}\pi r^3$

So when $2r = 50$

or $r = 25$cm

$$\frac{d}{dt} V = \frac{d}{dt} \frac{4}{3}\pi r^3$$

know $\frac{dV}{dt} = 100$, want $\cancel{\frac{d}{dt}}$ $\frac{dr}{dt}$ so

$$\frac{dV}{dt} = \cancel{\frac{4}{3}}\, 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{100}{4\pi r^2} \text{ if } r = 25$$

$$\frac{dr}{dt} = \frac{\cancel{100}}{\cancel{4}\pi (25)^2} = \boxed{\frac{1}{25\pi} \ cm/s}$$

3. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = k$ where k is a constant. Suppose that while compressing a certain gas, we have at a particular instant that the volume is 600 cm^3, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

t = time min
V = volume cm^3
P = pressure kPa

If $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$ kPA/min
what is $\frac{dV}{dt}$?

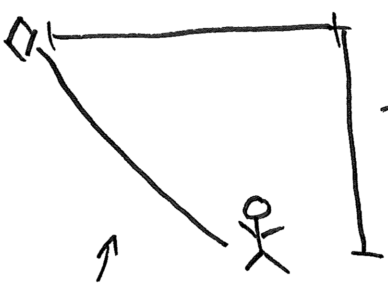
$PV = k$ so $\frac{dP}{dt}V + P\frac{dV}{dt} = 0$

so $\frac{dV}{dt} = \frac{dP/dt \cdot V}{-P} = \frac{20 \cdot 600}{-150}$

So $\boxed{\frac{dV}{dt} = -80 \text{ cm}^3/\text{min}}$

4. On a particularly windy day, a park-goer decides to fly a kite. Once their kite reaches an altitude of 300ft the wind at that height pulls the kite horizontally away from the kite-flier at a rate of 30 While doing so, the wind is pulling the kite horizontally away from the kite-flier at a rate of 44ft/sec.[1] How fast will the flier be releasing the string when the kite is ~~5~~ 400 ft away (from the park-goer)?

44 ft/sec



300ft

horizontally

altitude y
horiz distance x
string length s

change when 500 ft away

if $\frac{dx}{dt} = 44$, $y = 300$ and $x = \cancel{5}400$
what is $\frac{ds}{dt}$? and $\frac{dy}{dt} = 0$.

Relate the variables

$s^2 = x^2 + y^2$ so $2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

so $2x = \cancel{10}800$
$2y = 600$
$\frac{dx}{dt} = 44$
$\frac{dy}{dt} = 0.$

then $\frac{ds}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2s} = \frac{800 \cdot 44 + 0}{500}$

$2s = ? = 100$

$s^2 = \cancel{5}400^2 + \cancel{6}300^2 = \cancel{2500 + 3600 = 6100}$ 1600 + 900 = 2500

so $s = 500$

$\boxed{= \frac{8(44)}{5} \text{ ft/s}}$

[1] For those curious, this wind speed equates to a 6 of 12 on the Beaufort wind force scale. A 0 on the scale corresponds to essentially no wind while a 12 is hurricane force. A 6 on the scale would mean an umbrella is unpleasant to use.

Note: You are responsible for completing this portion.

5. Find the linearization of $f(x) = x^{3/4}$ at $a = 16$ use it to approximate $(17)^{3/4}$.

   Note: You need not find a decimal answer; if you do, you may be interested in comparing it against the approximation of $(17)^{3/4} \approx 8.37214402859$.

$f(x) = x^{3/4}$

$f'(x) = 3/4\, x^{-1/4}$

$a = 16$

$f(a) = \left((16)^{1/4}\right)^3 = 8$

$f'(a) = \frac{3}{4}(16)^{-1/4} = 3/8$

so $L(x) = 8 + \frac{3}{8}(x - 16)$

$= 8 + \frac{3x}{8} - 2 = \frac{3}{8}x + 6$

$\boxed{L(17) = \frac{3}{8}(17) + 6}$

6. (a) Linearize $f(x) = \ln x$ at $a = 1$ and use it to approximate $\ln 1.1$ and $\ln 2$.

   Note: You need not find a decimal answer; if you do, you may be interested in comparing it against the approximation of $\ln 1.1 \approx 0.0953101798043$.

$f(x) = \ln x$ so $a = 1$

$f'(x) = \frac{1}{x}$

$f(1) = 0$

$f'(1) = 1$

so $L(x) = 0 + 1(x - 1) = x - 1$

$L(1.1) = 1.1 - 1 = 0.1 \approx \ln(1.1)$

$L(2) = 2 - 1 = 1 \approx \ln(2)$

   (b) To make a linear approximation more accurate, we need take the point of tangency closer to the input we desire to approximate.

   Above you approximated $\ln 2$ using a linearization at $a = 1$. Use a linearization of $\ln x$ at $a = e$ to approximate 2.

$f(e) = 1$

$f'(e) = \frac{1}{e}$

$L(x) = 1 + \frac{1}{e}(x - e)$

$L(2) = \cancel{1 + \frac{1}{e} = \frac{e+1}{e} \approx}$

$= 1 + \frac{1}{e}(2 - e) = 1 + \frac{2}{e} - 1 = \frac{2}{e}$

   (c) Which approximation of $\ln 2$ was more accurate if $\ln 2 \approx 0.69314718056$? Why might this be?

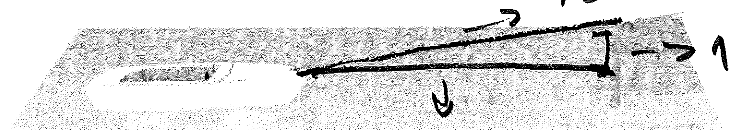
7. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.

$A = \pi r^2$. $\qquad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1m/s, how [redacted] fast is the area of the spill increasing when the radius is 30m?

$$\frac{dA}{dt} = 2\pi(30)\cdot 1 = 60\pi \ m^2/s$$

8. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 meter higher than the bow of the boat. If the rope is pulled in at a rate of 1 meter per second, how fast is the boat approaching the dock when it is 8 meters from the dock?



how fast is chang'g at x 8 meters?

dist. boat to dock x $\qquad$ if $\frac{dz}{dt} = 1$ m/s.

height of dock y

rope still out z $\qquad$ and $x = 8$ m $\quad$ what is $\frac{dx}{dt}$?

Note $y = 1$, $\frac{dy}{dt} = 0$. $\quad$ so $\quad z^2 = x^2 + 1 \rightarrow z^2 = x^2 + 1$

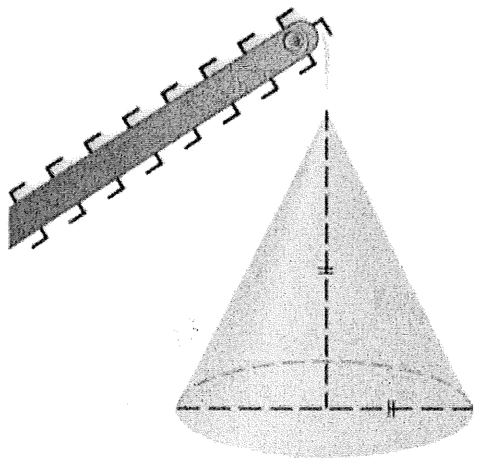
$x = 8 \rightarrow z = \sqrt{65}$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$ $\quad$ so $\quad \frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt}$ $\quad$ so $\boxed{\frac{dx}{dt} = \frac{\sqrt{65}}{8}}$

4

9. Gravel is being dumped from a conveyor belt at a rate of 10 ft^3/min. The coarseness is such that the gravel forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 11 ft high?

Hint: The formula for volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.



$V = \frac{1}{3}\pi r^2 h$ if $\frac{dV}{dt} = 10\ ft^3/min$ and $h = 11$, what is $\frac{dh}{dt}$?

$\cancel{\frac{dV}{dt} = \frac{2}{3}}$

$2r = h$ so $r = \frac{h}{2}$.

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h$ so $V = \frac{1}{12}\pi h^3$

$\frac{dV}{dt} = \cancel{3}\frac{1}{4}\pi h^2 \frac{dh}{dt} \Rightarrow 10 = \frac{1}{4}\pi \cdot 11^2 \frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = \frac{40}{\pi \cdot 11^2}}$$