Calc β, BRIDGE 2018 Name: Key Due: Monday, July 23

Homework #9: A few applications of the derivative

Note: This portion will be done in class.

1. Suppose an object's position s in space after t seconds (s) is given by

$$s(t) = t^3 - 6t^2 + 9t$$

where s is measured in meters (m) from the measurement point.

(a) How far is the object from the measurement point initially (at time $t = 0$)?

$s(0) = 0^3 - 6\cdot 0^2 + 9(0) = \boxed{0\text{ m}}$

At the measurement point.

(b) Find the velocity v of the object at t seconds.

$v(t) = \frac{ds}{dt} = \frac{\text{change in position}}{\text{change in time}}$

so $v(t) = s'(t) = 3t^2 - 12t + 9$

(c) What is the initial velocity of the object? after 2s? 4s?

Initial: $t=0$. $v(0) = 3\cdot 0^2 - 12(0) + 9 = \boxed{9\text{ m/s}}$ going forward

$t=2$ $v(2) = 3\cdot 4 - 12(2) + 9 = 12 - 24 + 9 = \boxed{-3\text{ m/s}}$ going backward

$t=4$ $v(4) = 3\cdot 16 - 12(4) + 9 = 48 - \cancel{24}\,48 + 9 = \boxed{9\text{ m/s}}$ back forward again

(d) When is the object at rest?

Not moving implies $\frac{ds}{dt} = 0$, so $v(t) = 0$.

$v(t) = 0$ if $3t^2 - 12t + 9 = 0$, so $t^2 - 4t + 3 = 0$

$\Rightarrow (t-3)(t-1) = 0$ so object at rest at $t = 1\text{ s}$ and $t = 3\text{ s}$

(e) Find the acceleration of the object at time t.

$a(t) = \frac{dv}{dt} = \frac{\text{change in velocity}}{\text{change in time}}$

so $a(t) = v'(t) = 6t - 12$

$v(t) = 3(t-3)(t-1)$ is pos. if $t > 3$ or $t < 1$

(f) Over what time span is the object speeding up? with pos. velocity.

Speeding up if $\frac{dv}{dt} = a(t) > 0$ and $v(t) > 0$.

Slowing down if $\frac{dv}{dt} = a(t) < 0$ and $v(t) > 0$

So $a(t) = 6t - 12 > 0$ if $6t > 12$ i.e. if $t > 2$

So $\boxed{\text{any time after 3 seconds}}$

(g) Sketch the graphs of s, v, a.

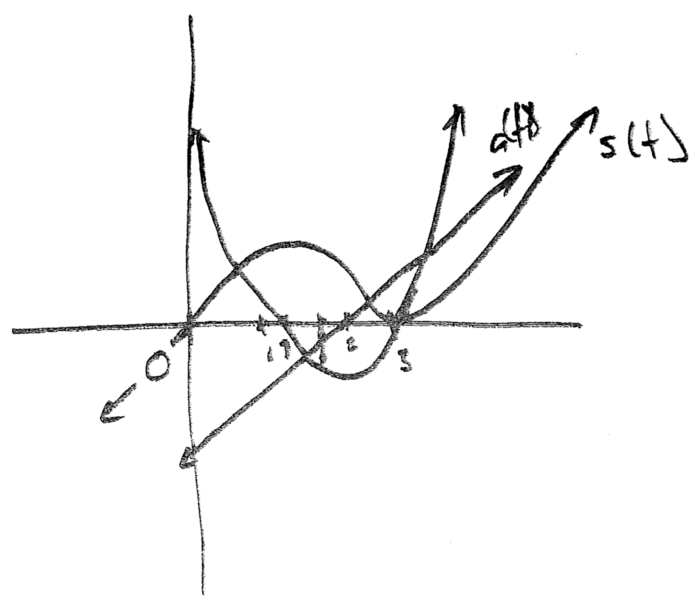
$s(t) = t^3 - 6t^2 + 9t$

$= t(t^2 - 6t + 9)$

$= t(t-3)^2$

$v(t) = 3(t-3)(t-1)$

$a(t) = 6t - 12$



2. The highest roller-coaster in the world *Kingda Ka* reaches a maximum height of 456 ft. Let's design a bigger one.

Suppose we wish the highest arc of the roller-coaster to reach 500 ft. Following a parabolic curve of $y = 500 - x^2$ would achieve this.

The issue is the descent: following this path, the track would crash to the ground only $x = \sqrt{500} = 22.36$ feet from the peak. To remedy this, let's follow another parabolic arc gently to the ground 40 ft from the peek. A parabola of the form $y = a(x-40)^2$ for some value of a should do the trick.

Determine a such that a passenger will enjoy a smooth descent on our roller coaster.

Hint: Find an a and x_0 such that

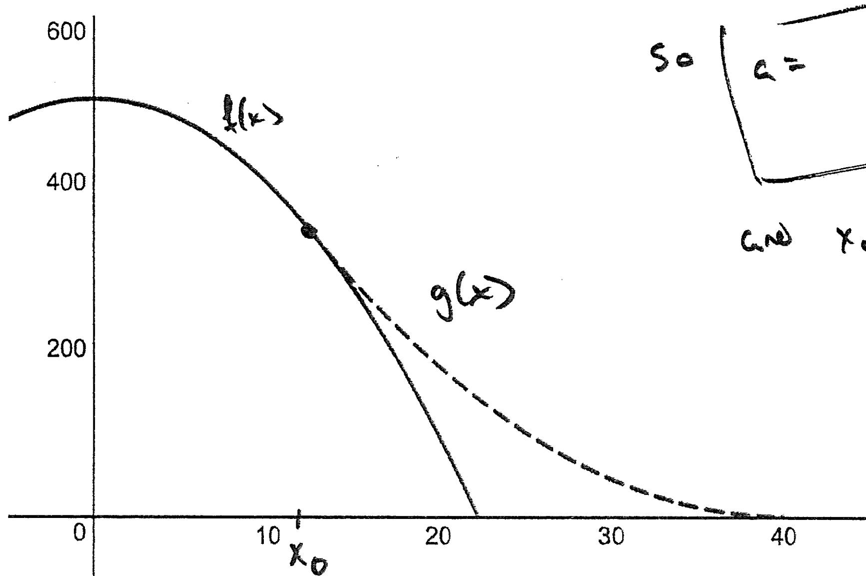
$$y = \begin{cases} 500 - x^2 & x \leqslant x_0 \\ a(x-40)^2 & x > x_0 \end{cases}$$

is both continuous and differentiable.

$a = \dfrac{-x_0}{x_0 - 40}$

So $\boxed{a = \dfrac{-12.5}{12.5 - 40}}$

and $x_0 = 12.5$.



First, ~~with~~ let $f(x) = 500 - x^2$, $g(x) = a(x-40)^2$.

Match derivatives @ x_0:

$f'(x) = -2x$

$g'(x) = 2a(x-40)$

so $f'(x_0) = g'(x_0) \Rightarrow -2x_0 = 2a(x_0 - 40)$

$\Rightarrow a = \dfrac{-x_0}{x_0 - 40}$

Match continuity @ x_0:

$f(x_0) = g(x_0) \Rightarrow 500 - x_0^2 = a(x_0 - 40)^2$

$\Rightarrow 500 - x_0^2 = \left(\dfrac{-x_0}{x_0 - 40}\right)(x_0 - 40)^2$

$\Rightarrow 500 - x_0^2 = -x_0^2 + 40x_0$

$\Rightarrow 500 = 40x_0$

$\Rightarrow \dfrac{50}{4} = x_0$

So $x_0 = \dfrac{25}{2} = 12.5$

3. Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in year 2020.

Let $P(t)$ = world pop. in millions ~~at~~ t years after 1950.

So $P(0)$ = "pop. in 1950" = 2560 mil.

$P(10)$ = "pop. in 1960" = 3040 mil.

Since growth rate is ~~cons~~ proportional to pop. size:

$$\frac{dP}{dt} = kP$$

thus for some k: $P(t) = P(0)e^{kt}$.

To find k, use $P(0) = 2560$ and $P(10) = 3040$.

Thus $P(10) = \underbrace{2560}_{P(0)} e^{k \cdot 10} = 3040$

so $e^{k \cdot 10} = \frac{3040}{2560}$

So $k = \frac{1}{10} \ln\left(\frac{304}{256}\right) \approx 0.017$ and $k \cdot 10 = \ln\left(\frac{3040}{2560}\right)$

$P(t) = 2560\, e^{0.017t}$ models the world population t years after 1950.

$k = 0.017$ is the relative growth rate:
"world pop. grows by about 1.7% annually."

$P(43) = 2560\, e^{0.017(43)} \approx 5360$ mil. is the est. pop in '93.

$P(70) = 2560\, e^{0.017(70)} \approx 8524$ mil. is the est. pop. in '20

Reality! $\frac{1993}{5588}$ so 4% error

in $\frac{2018}{7600}$ so 1.7% annual growth says $\frac{7860 \text{ in '20}}{80}$

Other applications of the derivative:

- In electromagnetism, *current* is the rate at which charge flows through a surface.

- Chemists use the derivative, to understand exactly the rate at which a chemical reaction is occurring; that is, the rate at which some combination of molecules turn into different molecules.

- Biologists can model the rate at which the flow of blood decreases as the cells approach the wall of a blood vessel, relative to pressure on either end of the vessel.

- In economics, the notions of marginal demand, marginal profit and marginal revenue are the derivatives of demand, profit and revenue respectfully.

- A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks.

- In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t. Of particular interest is the rate at which performance improves as time passes, that is, dP/dt.

- In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions).

- An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases.

- A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.

The underlying idea to each of these is, of course, the rate at which some quantity is changing. Abstracting away from each particular instance allows us to study the phenomenon of change independent of the idiosyncrasies implicit in each system we interact with. This is the power of mathematics: to see that one idea is informing many, many disparate phenomena in the world.

Note: You are responsible for completing this portion.

4. If a baseball is thrown upwards from the surface of Mars with an initial velocity of 10 meters per second, that it's position s in meters from the ground is given by

$$s(t) = 10t - \frac{93}{50}t^2$$

after t seconds.

(a) When will the ball hit the ground? (You do not need to find a decimal)

When $t > 0$ and $s(t) = 0$.

So $s(t) = 0 \Rightarrow 10t - \frac{93}{50}t^2 = 0$

$\Rightarrow 10 - \frac{93}{50}t = 0$ so $t = \frac{500}{93} \approx 5.38$ seconds.

(b) Find a function $v(t)$ which gives the velocity of the ball after t seconds.

$v(t) = s'(t) = 10 - \frac{93}{25}t$ m/s

(c) With what velocity will the rock hit the ground?

$v\left(\frac{500}{93}\right) = 10 - \frac{93}{25}\left(\frac{500}{93}\right) = 10 - \frac{500}{25}$

$= 10 - 20 = -10$ m/s.

(d) Find a function $a(t)$ which gives the acceleration of the ball after t seconds.

$a(t) = v'(t) = s''(t)$

$= -\frac{93}{25} \frac{m}{s^2}$ acceleration due to gravity.

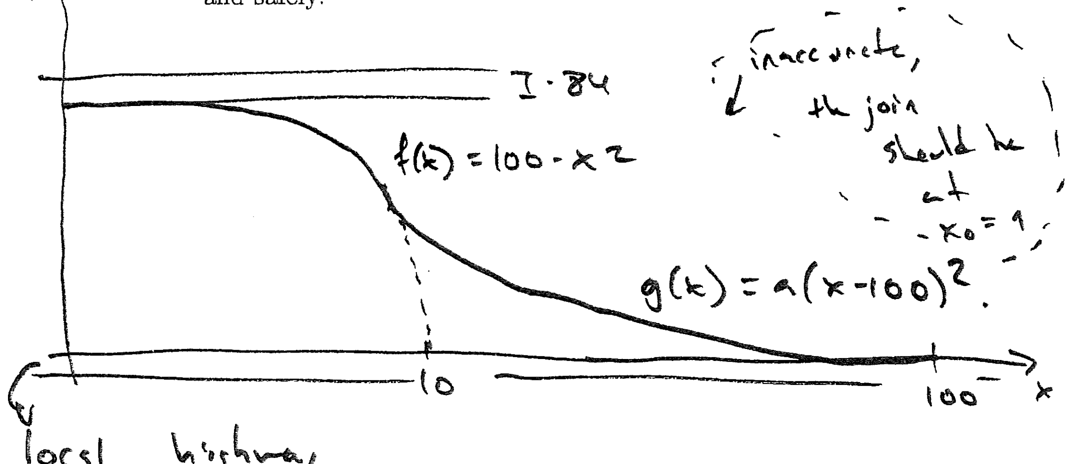
5. Due to popular demand, a new exit is to be installed on I-84 east of Storrs to help alleviate 195 traffic. The proposed plan seeks to connect to a local highway when I-84 parallels it only 100 meters north. Thus, the plan is to build an off-ramp which follows a parabolic arc $y = 100 - x^2$ where y is meters north of the local highway and x is meters east of the start of the off-ramp.

The connection to the local highway is to occur 100 meters east of the beginning of the off-ramp. With the current plan, the off-ramp would intersect the highway after only 10 meters. In addition, the linkage would be an extremely sharp turn causing slow down in traffic flow and the possibility of accidents if the motorists do not slow their vehicle appropriately.

To remedy this, we propose following a second parabolic arc $y = a(x-100)^2$ that joins up with the first after some distance x_0.

To ensure this linkage is safe the two curves will have to, of course, join continuously, but also in a differentiable manner. This will ensure the motorists do not experience a sudden jolt while following the path of the off-ramp.

Find an appropriate value of a and distance x_0 to ensure this expensive project concludes successfully and safely!



$f'(x) = -2x$

$g'(x) = 2a(x-100)$

let x_0 be where they join.

Derivatives: $f'(x_0) = g'(x_0) \Rightarrow -2x_0 = 2a(x_0 - 100)$ so $a = \frac{-x_0}{x_0 - 100}$.

Continuity: $f(x_0) = g(x_0) \Rightarrow 100 - x_0^2 = a(x_0 - 100)^2$

$a = \frac{-x_0}{x_0 - 100}$

$\Rightarrow 100 - \cancel{x_0^2} = -\cancel{x_0^2} + 100x_0$

so $x_0 = 1$.

$$\boxed{x_0 = 1 \Rightarrow a = \frac{-1}{1-100} = \frac{1}{99}}$$

thus $y = \begin{cases} 100 - x^2 & \text{if } x \le 1 \\ \frac{1}{99}(x-100)^2 & \text{if } x \ge 1 \end{cases}$

is the path of the off ramp.

6. The frequency of a vibrating violin string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density (mass divided by length).

(a) Find the rate of change of the frequency with respect to

i. the length (when T and ρ are constant),

$f(L) = \frac{1}{2L}\sqrt{T/\rho} = \frac{1}{2}L^{-1}\sqrt{T/\rho}$ so

$f'(L) = \frac{df}{dL} = \frac{-1}{2L^2}\sqrt{T/\rho}$

ii. the tension (when L and ρ are constant),

$f(T) = \frac{1}{2L}\sqrt{T}/\sqrt{\rho}$ so $\frac{df}{dT} = \frac{1}{2L}\frac{1}{2\sqrt{T\rho}}$ (since $(\sqrt{T})' = \frac{1}{2\sqrt{T}}$)

iii. and the linear density (when L and T are constant).

Here

$f(\rho) = \frac{1}{2L}\frac{\sqrt{T}}{\sqrt{\rho}}$ and ~~the~~ $\left(\frac{1}{\sqrt{\rho}}\right)' = (\rho^{-1/2})' = \frac{-1}{2}\rho^{-3/2}$ so $\frac{df}{d\rho} = \frac{-1}{4L}\sqrt{\frac{T}{\rho^3}}$

(b) The pitch of a note (how high or low the note sounds) is determined by the frequency f. (The higher the frequency, the more vibrations per second, thus the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note:

i. when the effective length of a string is decreased by placing a finger on the string (thus making a shorter portion of the string vibrate),

L decreases, $\frac{df}{dL}$ is negative

so f gets larger. Higher pitch

ii. when the tension is increased by turning a tuning peg.

T increases, $\frac{df}{dT}$ is pos.

so f gets larger Higher pitch

iii. and when the linear density is increased by switching to a heavier string.

ρ increases, $\frac{df}{d\rho}$ is ~~pos~~ neg,

so f gets smaller lower pitch