Calc β, BRIDGE 2018 Name: Key Due: Tuesday, July 17

Homework #8: Implicit differentiation

Note: Your work can only be assessed if it is legible.

1. Find $\frac{dy}{dx}$ using implicit differentiation. Solve for $\frac{dy}{dx}$ in terms of x and y in each case.

   (a) $2x^3 + x^2y - xy^3 = 2$

$$\xrightarrow{\frac{d}{dx}} 6x^2 + 2xy + x^2y' - y^3 - 3xy^2 \cdot y' = 0$$

$$\Rightarrow (x^2 - 3xy^2)y' = y^3 - 2xy - 6x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^3 - 2xy - 6x^2}{x^2 - 3xy^2}$$

   (b) $\cos(xy) = 1 + \sin y$

$$\xrightarrow{\frac{d}{dx}} -\sin(xy)\cdot(y + xy') = \cos(y)\cdot y'$$

$$\Rightarrow -\sin(xy)\cdot y - x\sin(xy)\cdot y' - \cos(y)\cdot y' = 0$$

$$\Rightarrow y'(-x\sin(xy) - \cos y) = \sin(xy)\cdot y$$

$$\Rightarrow \quad y' = \frac{\sin(xy)\cdot y}{-x\sin(xy)\cdot \cos y}$$

   (c) $e^y \sin x = x + xy$

$$\xrightarrow{\frac{d}{dx}} y'e^y\sin x + e^y\cos x = 1 + y + xy'$$

$$\Rightarrow y'(e^y\sin x - x) = 1 + y - e^y\cos x$$

$$\Rightarrow \quad y' = \frac{1 + y - e^y\cos x}{e^y\sin x - x}$$

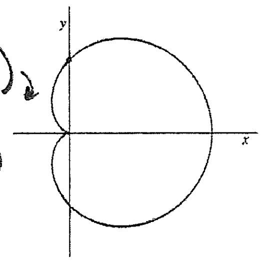
2. Use implicit differentiation to find an equation of the tangent line to the curve

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point $(0, 1/2)$.

Note: The graph of this equation is known as a cardioid (see below). This is not a graph of a function but we can still geometrically analyze it using implicit differentiation.

$\frac{dy}{dx}$: $2x + 2yy' = 2(2x^2 + 2y^2 - x) \cdot (4x + 4yy' - 1)$



So, point: $(0, 1/2)$, slope: 1

$\Rightarrow$ tangent line is

$$\boxed{y = x + 1/2}$$

So

$$2x + 2yy' = 16x^3 + 16xy^2 - 8x^2 - 4x^2 - 4y^2 + 2x + y'(16x^2y + 16y^3 - 8xy)$$

$$\Rightarrow y'(2y - 16x^2y - 16y^3 + 8xy) = 16x^3 + 16xy^2 - 12x^2 - 4y^2,$$

at $(x, y) = (0, 1/2)$ $\Rightarrow$ $y'(2(1/2) - 0 - 2 + 0) = 0 + 0 - 0 - 4(1/2)^2$

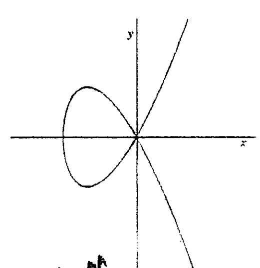
$\Rightarrow$ $-y' = -1$ $\Rightarrow$ $y' = 1.$ $\leftarrow$ slope of tangent at $(0, 1/2)$

3. The curve with equation

$$y^2 = x^3 + 3x^2$$

is called the Tschirnhuasen cubic (see below). At what points does this curve have horizontal tangents?

Horizontal tangent: $\frac{dy}{dx} = 0$.



Here $\frac{dy}{dx}$ is found by imp. diff.

$y^2 = x^3 + 3x^2$ $\Rightarrow$ $2yy' = 3x^2 + 6x$

$\Rightarrow$ $y' = \dfrac{3x^2 + 6x}{2y}$.

So $y' = 0$ if $3x^2 + 6x = 0$ and $2y \neq 0$. $\Rightarrow$ $x^2 + 2x = 0$ $\Rightarrow$ $x(x+2) = 0$

So $x = 0$ or $x = -2$.

if $x = 0$, $y^2 = 0^3 + 3(0)^2 = 0$ and y' is undefined.

if $x = -2$, $y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$.

So y' is defined and equals 0 if $x = -2$ and $y^2 = 4$.

The points are $(-2, -2)$ and $(-2, 2)$

In class we used implicit differentiation to find derivatives of a couple of the inverse functions in this table.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\ln x$	$\frac{1}{x}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\log_a x$	$\frac{1}{(\ln a)x}$
$\arctan x$	$\frac{1}{1+x^2}$		

Here you will use implicit differentiation to find the rest.

4. *Inverse trig. functions.* Simplify your answers.

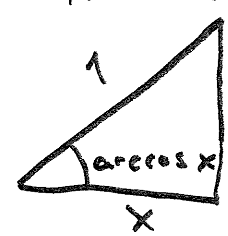
   (a) Use implicit differentiation to find the derivative of $y = \arccos x$.

$y = \arccos x \Rightarrow \cos y = x$

$y = \arccos x$ so $\sin(y) = \sin(\arccos x) = \sqrt{1-x^2}$

thus

$\Rightarrow (-\sin y)\cdot y' = 1$



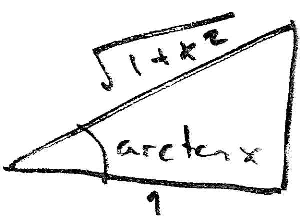
$\Rightarrow y' = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}$.

$y' = \frac{-1}{\sqrt{1-x^2}}$

   (b) Use implicit differentiation to find the derivative of $y = \arctan x$.

Here $\tan y = x$. So $y' \cdot \sec^2 y = 1 \Rightarrow y' = \frac{1}{\sec^2(\arctan x)}$.

$= \frac{1}{(\sec(\arctan x))^2}$



$\Rightarrow \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$

so

$\sec(\arctan x) = \sqrt{1+x^2} \longrightarrow \boxed{y' = \frac{1}{1+x^2}}$

5. Use implicit differentiation to find the derivative of $y = \log_a x$.

Here $a^y = x$ so $\ln a \cdot a^y \cdot y' = 1$

$\Rightarrow y' = \frac{1}{\ln a \cdot a^y}$ since $y = \log_a x$, $a^y = a^{\log_a x} = x$

$\Rightarrow \boxed{y' = \frac{1}{x \cdot \ln a}}$

6. In class, we used logarithmic differentiation to show that for any real number n, $(x^n)' = nx^{n-1}$. Use that same technique to find the derivative of the following functions.

(a) $y = \sqrt{x}^{\,x}$

$$y = (\sqrt{x})^x \Rightarrow \ln y = x \ln(\sqrt{x}) \Rightarrow \ln y = \frac{x}{2} \ln x$$

$$\xrightarrow{d/dx} \frac{y'}{y} = \frac{1}{2}\ln x + \frac{x}{2}\cdot\frac{1}{x} \quad \text{so} \quad \frac{y'}{(\sqrt{x})^x} = \frac{1}{2}(\ln x + 1)$$

$$\Rightarrow y' = \frac{(\sqrt{x})^x}{2}(\ln(x) + 1)$$

(b) $y = x^{\cos x}$

$$\ln y = \cos x \ln x \Rightarrow \frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}$$

$$\Rightarrow y' = y\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

$$\Rightarrow y' = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

7. Differentiate the following functions. You may use any rule or identity.

(a) $y = \ln x^2$

Chain-rule: $y' = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$.

Log-rule: $y = 2\ln x$ so $y' = \frac{2}{x}$.

(b) $f(x) = \frac{1}{x^3}$

$f(x) = x^{-3}$ so $f'(x) = -3x^{-4}$.

(c) $f(x) = \frac{1}{\sqrt[3]{x}}$

$f(x) = x^{-1/3}$ so $f'(x) = \frac{-1}{3} x^{-1/3 - 1}$

$= \frac{-1}{3} x^{-4/3} = \boxed{\frac{-1}{3\sqrt[3]{x^4}}}$

(d) $y = \log_2(\arctan x)$

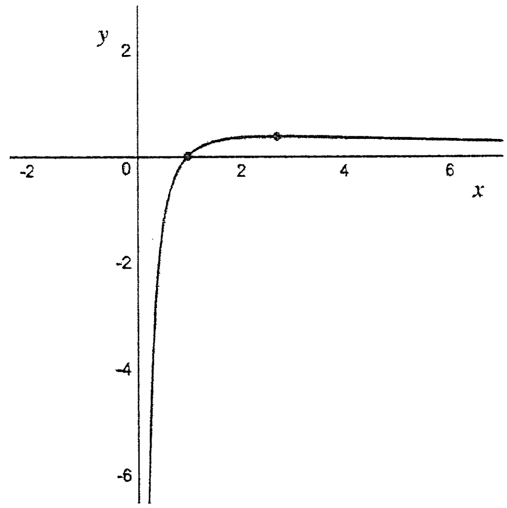
$y' = \frac{1}{\ln 2\,(\arctan x)} \cdot \frac{1}{1+x^2}$.

(e) $f(x) = x \ln x - x$

$f'(x) = \ln x + x\left(\frac{1}{x}\right) - 1$

$= \ln x + 1 - 1 = \boxed{\ln x}$

8. Here is a graph of the function $y = \dfrac{\ln x}{x}$.



Find equations of the tangent lines to this graph at:

(a) $x = 1$

Point: $x = 1 \Rightarrow y = \dfrac{\ln 1}{1} = 0$ so $(1, 0)$

Slope: $y' = \left(\dfrac{\ln x}{x}\right)' = \left(\ln x \cdot x^{-1}\right)' = \dfrac{1}{x} \cdot x^{-1} + \ln x \cdot (-x^{-2})$

$$= \frac{1}{x^2} + \frac{-\ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

so $y'(1) = \dfrac{1 - \ln 1}{1^2} = 1$

$\Rightarrow$ tangent line: $\boxed{y = (x - 1)}$

(b) and $x = e$.

Point: $x = e \Rightarrow y(e) = \dfrac{\ln e}{e} = \dfrac{1}{e}$.

Slope: $x = e \Rightarrow y'(e) = \dfrac{1 - \ln(e)}{e^2} = \dfrac{1 - 1}{e^2} = 0$

so tangent line is

$$\left(y - \frac{1}{e}\right) = 0(x - e)$$

$\Rightarrow \boxed{y = 1/e}$