Calc $\beta$, BRIDGE 2018 Name: Key Due: Thursday, July 12

# Homework #6: The algebraic properties of the derivative

*Note:* Your work can only be assessed if it is legible. A single final answer will be considered a guess: minimal credit will be given without justification of your conclusions. In class we have verified the power rule for $n$ a positive integer, the sum rule, and the product rule. Therefore, you may use those rules freely. Any other rule of differentiation you wish to use must first be verified before being invoked.

1. In class, we mentioned that for any constant function $f(x) = c$, the derivative $f'(x) = 0$. Prove this fact using the limit definition of the derivative.

$f(x) = c$

Proof. $$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0. \checkmark$$

2. Find the derivative of $f(x) = x^{3/2}$.

*Note:* The power rule discussed in class does not apply here as 3/2 is not a positive integer. [redacted])

$$\frac{d}{dx}\left(x^{3/2}\right) = \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} \cdot \frac{(x+h)^{3/2} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h\left((x+h)^{3/2} + x^{3/2}\right)} = \lim_{h\to 0} \frac{x^3 + 3xh^2 + 3x^2h + h^3 - x^3}{h\left((x+h)^{3/2} + x^{3/2}\right)}$$

$$= \lim_{h\to 0} \frac{3xh + h^2 + 3x^2}{(x+h)^{3/2} + x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \boxed{\frac{3}{2} x^{1/2}}$$

Or -

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \cdot \lim_{h\to 0} \frac{1}{(x+h)^{3/2} + x^{3/2}}$$

$$= (x^3)' \cdot \frac{1}{2x^{3/2}}$$

$$= \frac{3x^2}{2x^{3/2}} = \frac{3}{2} x^{1/2}$$

3. In class, we mentioned that for $c$ a constant

$$\frac{d}{dx}(cf(x)) = c\left(\frac{d}{dx}f(x)\right).$$

Prove this fact using the limit definition of the derivative.

Proof.

$$\frac{d}{dx} cf(x) = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}$$

limit law

$$= \lim_{h\to 0} c\left(\frac{f(x+h) - f(x)}{h}\right) = c\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= cf'(x).$$

don't need.

4. Find the derivative of $f(x) = 6x^{3/2} + 2x^2 + 3$.

By problem 1,2,3, the power rule and the ~~[illegible]~~ sum rule.

$$f'(x) = 6\left(x^{3/2}\right)' + 2\left(x^2\right)' + (3)'$$

$$= 6\left(\tfrac{3}{2}x^{1/2}\right) + 2(2x) + 0$$

$$= \boxed{9x^{1/2} + 4x}$$

5. Consider the difference rule for differentiation:

$$(f(x) - g(x))' = f'(x) - g'(x).$$

proof.

(a) Verify this rule using the limit definition of the derivative.

$$(f(x)-g(x))' = \lim_{h \to 0} \frac{(f(x+h)-g(x+h))-(f(x)-g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \frac{(g(x+h)-g(x))}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x) - g'(x).$$ ✓

(b) Reverify this fact using only the facts that

$$(f(x) + g(x))' = f'(x) + g'(x) \text{ and } (cf(x))' = c(f(x))' = cf'(x).$$

Proof.

*Hint:* You can rewrite $f(x) - g(x)$ as $f(x) + (-1)g(x)$.

$$(f(x)-g(x))' = (f(x)+(-1)g(x))'$$

sum rule $\Rightarrow$ $$= f'(x) + ((-1)g(x))'$$

$$= f'(x) + (-1)(g(x))' = f'(x) - g'(x).$$ ✓

6. Replicate the proof of the product rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

given in class.

Proof.

$$\frac{d}{dx}(f(x)\cdot g(x)) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) + \overbrace{f(x+h)g(x) - f(x+h)g(x)}^{=0} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right) + g(x)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$$



$$= f(x)g'(x) + f'(x)g(x).$$ ✓

7. Following a similar strategy as in the previous question, verify the quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Hint: Begin by simplifying

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

with a common denominator and then add zero in the form $0 = -f(x)g(x) + f(x)g(x)$.

Proof.

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x)\underbrace{- f(x)g(x) + f(x)g(x)}_{=0} - f(x)g(x+h)}{g(x+h)g(x)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\left(\frac{f(x+h)-f(x)}{g(x+h)}\right) - f(x)\left(\frac{g(x+h)-g(x)}{g(x+h)g(x)}\right)\right)$$

$$= \left[\lim_{h\to 0} \frac{1}{g(x+h)} \cdot \left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)\right]$$

$$- \left[\lim_{h\to 0} \frac{f(x)}{g(x+h)g(x)}\right] \cdot \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right]$$

$$= \frac{1}{g(x)} \cdot f'(x) - \frac{f(x)}{g(x)\cdot g(x)} \cdot g'(x) = \frac{f'(x)g(x)}{(g(x))^2} - \frac{f(x)g'(x)}{(g(x))^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

8. In class we mentioned that $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$.

   Verify the first claim: $\dfrac{d}{dx}\sin x = \cos x$.

   *Hint:* You may use these three facts without justification

   i. $\sin(x+h) = \sin x \cos h + \cos x \sin h$

   ii. $\displaystyle\lim_{t\to 0}\frac{\sin t}{t} = 1$

   iii. $\displaystyle\lim_{t\to 0}\frac{\cos t - 1}{t} = 0$

Proof.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h} + \sin x \cdot \frac{\cos h - 1}{h}\right)$$

$$= \cos x \cdot \left(\lim_{h\to 0}\frac{\sin h}{h}\right) + \sin x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right)$$

$$= \cos x \cdot (1) + \sin x \cdot (0)$$

$$= \cos x \checkmark$$

9. We can use similar identities to show that $\frac{d}{dx}\cos x = -\sin x$.

   Use this fact, the result of the last question, and the quotient rule to verify that

$$\frac{d}{dx}\tan x = \sec^2 x.$$

Proof.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \qquad f = \sin x,\ g = \cos x \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{(\sin x)'\cdot \cos x - (\sin x)(\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x. \checkmark$$